MATH 333: Partial Differential Equations

Problem Set 1, Final version

Due Date: Wed., Sept. 13, 2006

Note: Certain of these "problems" do not require write-ups.

1. Look over the course webpage. In particular, read through the course syllabus, get accustomed to requesting a course calendar, and look over the interface for getting homework assignments.

2. Please email me a note at `scofield@calvin.edu`. In the note, please provide your name, your major, a list of the mathematics courses you have taken, your computing courses (it's better if you tell me what the course had you doing than it is to give a course name) and experiences, your reasons for taking MATH 333, and anything else about yourself you think would be helpful or interesting. Also, is there anything specific I can include in my prayers concerning you?

3. Read Sections 1.1–1.2. Make every attempt to understand the examples and fill in the details. You are welcome to bring those things you do not understand up in class.

4. Do Exercise 1.1. In addition to parts (a)-(c), also classify the differential equations by their order.

5. Do Exercise 1.2.

6. Download and install a copy of Octave on your personal computer. (Windows users who do not already have *cygwin* installed on your computer—and you know it if you do—should choose the *Octave-Forge Files* link.) To run Octave (after installing it) in a unix/linux/macintosh system, you likely can type `octave` in a terminal window. I believe in a Microsoft Windows system you will have an icon somewhere off the "Start" menu that says "Octave". Once you have it running, take a look at this .pdf file providing a general tutorial of Octave. Go through the first 5 sections (up to "Octave programming I: Script files") of the tutorial, executing the commands in your own Octave window as you go, in order to get an initial sense of the software interface Octave provides. I suspect that we may begin to have problems with Octave installations in Section 5 ("Plotting graphs"). If you are having difficulties, determine what they are as specifically as possible, bring them up in class, and we will see how common such difficulties are and/or whether someone has figured out a workaround.

7. Read through Project 1.2 as far as the top of p. 35 (the bullet that precedes problem (m)). Do exercises (h), (i) and (n). You may use **Octave** for part (n). Pertinent commands you might wish to find out more about include `eig()` and `dot()`. You can get help for any built-in command—for instance, `eig()`—by typing

```
> help eig
```

8. Functions of the form x^r, where r is a (fixed) real number are called "power functions". When $r \geq 0$, the resulting power function is continuous on all of $[0, 1]$, and using some of the theorems of Calculus, it is easy to establish that the function is, therefore, a member of $L^2(0, 1)$. But what of the power functions with $r < 0$? They are continuous on the open interval $(0, 1)$, but not on the closed interval $[0, 1]$. Are any of them members of $L^2(0, 1)$? For which choices of $r < 0$? Note that, for $r < 0$,

$$\int_0^1 x^{2r}\, dx$$

is an improper integral.

9. We say that vectors $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^n$ are *orthogonal* if

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle \;:=\; \sum_{k=1}^{n} u_k v_k \;=\; 0.$$

This same concept of orthogonality holds for functions in $L^2(a, b)$ using the inner product we defined as

$$\langle f, g \rangle \;:=\; \int_a^b f(x) g(x)\, dx.$$

Consider two sine functions $\sin(n\pi\cdot)$ and $\sin(m\pi\cdot)$, where m, n are both positive integers with $m \neq n$. Show that, as functions in $L^2(0, 1)$ (that is, under the inner product of $L^2(0, 1)$), these functions are orthogonal (independent of the choices of m and n). Hint: You will likely find useful the trigonometric identities

$$\cos(\phi \pm \psi) \;=\; \cos\phi \cos\psi \mp \sin\phi \sin\psi,$$

which together imply

$$\sin\phi \sin\psi \;=\; \frac{1}{2}\left[\cos(\phi - \psi) - \cos(\phi + \psi)\right].$$

(Do you see why?) Are these sine functions still orthogonal when considered as functions in $L^2(-1, 1)$? How about $L^2(0, 1/2)$?

10. (a) Below are three pairs of vectors in $\mathbb{R}^2$, given using *Octave* notation.

$$\begin{array}{ll} \boldsymbol{u}_1 = [3; 4], & \boldsymbol{u}_2 = [-4; 3] \\ \boldsymbol{u}_1 = [1; 0], & \boldsymbol{u}_2 = [0; -2] \\ \boldsymbol{u}_1 = [1; 0], & \boldsymbol{u}_2 = [1; 1] \end{array}$$

For each different pair of vectors, do the following

(i) Determine the lengths (norms) of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$.

(ii) Write, if possible, the vector $\boldsymbol{v} = [-8; 5]$ as a linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$—that is, find scalars c_1, c_2 such that

$$\boldsymbol{v} \;=\; c_1 \boldsymbol{u}_1 + c_2 + \boldsymbol{u}_2.$$

(iii) Determine the inner products of $\boldsymbol{v}$ (given above) with $\boldsymbol{u}_1$, $\boldsymbol{u}_2$.

(iv) Indicate whether your answers to parts (i) and (iii) might aid in answering part (ii). Why is the answer sometimes *yes* and sometimes *no*?

(b) In Calculus we learn about the vector projection of one vector $\boldsymbol{v}$ onto another vector $\boldsymbol{u}$, given by

$$\text{proj}_{\boldsymbol{u}}\boldsymbol{v} \ := \ \frac{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}{\|\boldsymbol{u}\|^2}\,\boldsymbol{u}.$$

How is what you were doing in part (a) related to this concept?

(c) Projections are fundamental in the function space $L^2(a,b)$ as well:

$$\text{proj}_f g \ := \ \frac{\langle g, f \rangle}{\|f\|^2} f.$$

Suppose we are working in $L^2(0,1)$ and $f(\cdot) = \sin(n\pi\cdot)$ for some integer $n \geq 1$. Show that, no matter the choice of n,

$$\frac{\langle g, f \rangle}{\|f\|^2} \ = \ 2\int_0^1 g(x)\sin(n\pi x)\,dx.$$

Hint: A helpful trigonometric identity here is

$$\sin^2\phi \ = \ \frac{1}{2}\left[1 - \cos(2\phi)\right].$$

11. We asserted in class (without proof) that the sup norm

$$\|f\|_\infty \ := \ \sup_{x\in(a,b)} |f(x)|$$

is a norm on $\mathcal{C}(a,b)$. As we have seen, there are 4 requirements placed on $\|\cdot\|_\infty$ for it to be a *norm*, one of which is the triangle inequality

$$\|f+g\|_\infty \ \leq \ \|f\|_\infty + \|g\|_\infty, \qquad \text{for all } \ f, g \in \mathcal{C}(a,b).$$

You need not show that this inquality is satisfied, though it is a worthwhile exercise to think about why this is so. For this exercise, you are to find two functions (define them explicitly, along with the interval (a,b) you are using) so that

(a) strict inequality holds. That is, find f_1, $g_1 \in \mathcal{C}(a,b)$ with

$$\|f_1 + g_1\|_\infty \ < \ \|f_1\|_\infty + \|g_1\|_\infty.$$

(b) equality holds. That is, find f_2, $g_2 \in \mathcal{C}(a,b)$ with

$$\|f_2 + g_2\|_\infty \ = \ \|f_2\|_\infty + \|g_2\|_\infty.$$