1 Determinants and the Solvability of Linear Systems

In the last section we learned how to use Gaussian elimination to solve linear systems of n equations in n unknowns. The section completely side-stepped one important question: that of whether a system has a solution and, if so, whether it is unique.

Consider the case of two lines in the plane

$$\begin{aligned} ax + by &= e \\ cx + dy &= f. \end{aligned} \tag{1}$$

In fact, we know that intersections between two lines can happen in any of three different ways:

1. the lines intersect at a unique point (i.e., solution exists and is unique),
2. the lines are coincident (that is, the equations represent the same line and there are infinitely many points of intersection; in this case a solution exists, but is not unique), or
3. the lines are parallel but not coincident (so that no solution exists).

Experience has taught us that it is quite easy to decide which of these situations we are in before ever attempting to solve a linear system of two equations in two unknowns. For instance, the system

$$\begin{aligned} 3x - 5y &= 9 \\ -5x + \frac{25}{3}y &= -15 \end{aligned}$$

obviously contains two representations of the same line (since one equation is a constant multiple of the other) and will have infinitely many solutions. In contrast, the system

$$\begin{aligned} x + 2y &= -1 \\ 2x + 4y &= 5 \end{aligned}$$

will have no solutions. This is the case because, while the left sides of each equation — the sides that contain the coefficients of x and y which determine the slopes of the lines — are in proportion to one another, the right sides are not in the same proportion. As a result, these two lines will have the same slopes but not the same y-intercepts. Finally, the system

$$\begin{aligned} 2x + 5y &= 11 \\ 7x - y &= -1. \end{aligned}$$

will have just one solution (one point of intersection), as the left sides of the equations are not at all in proportion to one another.

What is most important about the preceding discussion is that we can distinguish situation 1 (the lines intersecting at one unique point) from the others simply by looking at the coefficients a, b, c and d from equation (1). In particular, we can determine the ratios $a : c$

and $b : d$ and determine whether these ratios are the same or different. Equivalently, we can look at whether the quantity

$$ad - bc$$

is zero or not. If $ad - bc \neq 0$ then the system has one unique point of intersection, but if $ad - bc = 0$ then the system either has no points or infinitely many points of intersection. If we write equation (1) as a matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix},$$

we see that the quantity $ad - bc$ is dependent only upon the coefficient matrix. Since this quantity "determines" whether or not the system has a unique solution, it is called the *determinant* of the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and is sometimes abbreviated as $\det(\mathbf{A})$, $|\mathbf{A}|$ or

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

While it is quite easy for us to determine in advance the number of solutions which arise from a system of two linear equations in two unknowns, the situation becomes a good deal more complicated if we add another variable and another equation. The solutions of such a system

$$\begin{aligned} ax + by + cz &= l \\ dx + ey + fz &= m \\ ex + hy + kz &= n \end{aligned} \tag{2}$$

can be thought of as points of intersection between three planes. Again, there are several possibilities:

1. the planes intersect at a unique point,
2. the planes intersect along a line,
3. the planes intersect in a plane, or
4. the planes do not intersect.

It seems reasonable to think that situation 1 can once again be distinguished from the other three simply by performing some test on the numbers a, b, c, d, e, f, g, h and k. As in the case of the system (1), perhaps if we write system (2) as the matrix equation

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l \\ m \\ n \end{bmatrix},$$

we will be able to define an appropriate quantity $\det(\mathbf{A})$ that depends only on the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

in such a way that, if $\det(\mathbf{A}) \neq 0$ then the system has a unique solution (situation 1), but if $\det(\mathbf{A}) = 0$ then one of the other situations (2–4) is in effect.

Indeed it is possible to define $\det(\mathbf{A})$ for a square matrix $\mathbf{A}$ of arbitrary dimension. For our purposes, we do not so much wish to give a rigorous definition of such a determinant as we do wish to be able to find it. As of now, we do know how to find it for a 2×2 matrix:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

For square matrices of dimension larger than 2 we will find the determinant using *cofactor expansion.*

Let $\mathbf{A} = (a_{ij})$ be an arbitrary $n \times n$ matrix; that is,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

We define the (i, j)-*minor* of $\mathbf{A}$, M_{ij}, to be the determinant of the matrix resulting from crossing out the i^{th} row and the j^{th} column of $\mathbf{A}$. Thus, if

$$\mathbf{B} = \begin{bmatrix} 1 & -4 & 3 \\ -3 & 2 & 5 \\ 4 & 0 & -1 \end{bmatrix},$$

we have nine possible minors M_{ij} of $\mathbf{B}$, two of which are

$$M_{21} = \begin{vmatrix} -4 & 3 \\ 0 & -1 \end{vmatrix} = 4 \qquad \text{and} \qquad M_{33} = \begin{vmatrix} 1 & -4 \\ -3 & 2 \end{vmatrix} = -10.$$

A concept that is related to the (i, j)-minor is the (i, j)-*cofactor*, C_{ij}, which is defined to be

$$C_{ij} := (-1)^{i+j} M_{ij}.$$

Thus, the matrix $\mathbf{B}$ above has 9 cofactors C_{ij}, two of which are

$$C_{21} = (-1)^{2+1} M_{21} = -4 \qquad \text{and} \qquad C_{33} = (-1)^{3+3} M_{33} = -10.$$

Armed with the concept of cofactors, we are prepared to say how the determinant of an arbitrary square matrix $\mathbf{A} = (a_{ij})$ is found. It may be found by expanding in cofactors along the i^{th} row:

$$\det(\mathbf{A}) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik};$$

or, alternatively, it may be found by expanding in cofactors along the j^{th} column:

$$\det(\mathbf{A}) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{k=1}^{n} a_{kj}C_{kj}.$$

Thus, $\det(\mathbf{B})$ for the 3×3 matrix $\mathbf{B}$ above is

$$\begin{aligned}
\det(\mathbf{B}) = \begin{vmatrix} 1 & -4 & 3 \\ -3 & 2 & 5 \\ 4 & 0 & -1 \end{vmatrix} &= 4(-1)^{3+1}\begin{vmatrix} -4 & 3 \\ 2 & 5 \end{vmatrix} \\
&\quad +(0)(-1)^{3+2}\begin{vmatrix} 1 & 3 \\ -3 & 5 \end{vmatrix} + (-1)(-1)^{3+3}\begin{vmatrix} 1 & -4 \\ -3 & 2 \end{vmatrix} \\
&= 4(-20-6) + 0 - (2-12) \\
&= -94.
\end{aligned}$$

Here we found $\det(\mathbf{B})$ via a cofactor expansion along the third row. You should verify that a cofactor expansion along any of the other two rows would also lead to the same result. Had we expanded in cofactors along one of the columns, for instance column 2, we would have

$$\begin{aligned}
\det(\mathbf{B}) &= (-4)(-1)^{1+2}\begin{vmatrix} -3 & 5 \\ 4 & -1 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix} + (0)(-1)^{3+2}\begin{vmatrix} 1 & 3 \\ -3 & 5 \end{vmatrix} \\
&= 4(3-20) + 2(-1-12) + 0 \\
&= -94.
\end{aligned}$$

This process can be used iteratively on larger square matrices. For instance

$$\begin{aligned}
\begin{vmatrix} 3 & 1 & 2 & 0 \\ -1 & 0 & 5 & -4 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & -3 & 1 \end{vmatrix} &= (0)(-1)^{4+1}\begin{vmatrix} 1 & 2 & 0 \\ 0 & 5 & -4 \\ 1 & 0 & -1 \end{vmatrix} + (0)(-1)^{4+2}\begin{vmatrix} 3 & 2 & 0 \\ -1 & 5 & -4 \\ 1 & 0 & -1 \end{vmatrix} \\
&\quad +(-3)(-1)^{4+3}\begin{vmatrix} 3 & 1 & 0 \\ -1 & 0 & -4 \\ 1 & 1 & -1 \end{vmatrix} + (1)(-1)^{4+4}\begin{vmatrix} 3 & 1 & 2 \\ -1 & 0 & 5 \\ 1 & 1 & 0 \end{vmatrix},
\end{aligned}$$

where our cofactor expansion of the original determinant of a 4×4 matrix along its fourth row expresses it in terms of the determinants of several 3×3 matrices. We may proceed to find these latter determinants using cofactor expansions as well:

$$\begin{aligned}
\begin{vmatrix} 3 & 1 & 0 \\ -1 & 0 & -4 \\ 1 & 1 & -1 \end{vmatrix} &= (3)(-1)^{1+1}\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} + (1)(-1)^{1+2}\begin{vmatrix} -1 & -4 \\ 1 & -1 \end{vmatrix} + (0)(-1)^{1+3}\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} \\
&= 3(0+4) - (1+4) + 0 \\
&= 7,
\end{aligned}$$

where we expanded in cofactors along the first row, and

$$\begin{aligned}
\begin{vmatrix} 3 & 1 & 2 \\ -1 & 0 & 5 \\ 1 & 1 & 0 \end{vmatrix} &= (1)(-1)^{1+2}\begin{vmatrix} -1 & 5 \\ 1 & 0 \end{vmatrix} + (0)(-1)^{2+2}\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} + (1)(-1)^{3+2}\begin{vmatrix} 3 & 2 \\ -1 & 5 \end{vmatrix} \\
&= -(0-5) + 0 - (15+2) \\
&= -12,
\end{aligned}$$

where this cofactor expansion was carried out along the second column. Thus

$$\begin{vmatrix} 3 & 1 & 2 & 0 \\ -1 & 0 & 5 & -4 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & -3 & 1 \end{vmatrix} = 0 + 0 + (3)(7) + (1)(-12) = 9.$$

The matrices whose determinants we computed in the preceding paragraph are the coefficient matrices for the two linear systems

$$\begin{aligned}
x_1 - 4x_2 + 3x_3 &= b_1 \\
-3x_1 + 2x_2 + 5x_3 &= b_2 \\
4x_1 - x_3 &= b_3
\end{aligned}$$

and

$$\begin{aligned}
3x_1 + x_2 + 2x_3 &= b_1 \\
-x_1 + 5x_3 - 4x_4 &= b_2 \\
x_1 + x_2 - x_4 &= b_3 \\
-3x_3 + x_4 &= b_4
\end{aligned}$$

respectively; that is, if we write each of these systems as a matrix equation of the form $\mathbf{Ax} = \mathbf{b}$, with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix},$$

then the coefficient matrix $\mathbf{A}$ in each case is one for which we have already computed the determinant and found it to be nonzero. This means that no matter what values are used for b_1, b_2, b_3 (and b_4 in the latter system) there is exactly one solution for the unknown vector $\mathbf{x}$.