4 Vector Calculus

4.1 Parametrized Curves and Surfaces

Rigorous treatment of both curves and surfaces necessarily requires the idea of parametrization. Parametrized curves are first encountered by MATH 161 students in Section 3.5 of University Calculus, by Hass, Weir, and Thomas (particularly pp. 170–173). The authors of that text revisit the idea, in greater depth, in Chapter 11, most importantly in Section 1 of that chapter. These may be useful sections to consult along with reading this section. The vector functions that arise as solutions to (linear) first-order systems of differential equations (see, for instance, Chapter 7 of Elementary Differential Equations, by Boyce and DiPrima) are themselves parametrized curves.

Parametrizations of surfaces are generally not treated in the first year of calculus, so the concept will likely be new. However, as we will see, parametrized surfaces have much in common with parametrized curves, differing mainly in the fact that parametrized surfaces have two degrees of freedom (two parameters) whereas parametrized curves have just one.

4.1.1 Parametrized curves

A parametrization of a curve (or parametrized curve) in $\mathbb{R}^n$ is a continuous mapping $r: I \to \mathbb{R}^n$, where $I$ is an interval in $\mathbb{R}$, whose range is the curve in question. Note that for each $t \in I$, $r(t)$ is a vector in $\mathbb{R}^n$, so a parametrized curve is sometimes called a vector function. Such a function $r$ has $n$ components, and may be written as

$$r(t) = x_1(t) \mathbf{e}_1 + x_2(t) \mathbf{e}_2 + \cdots + x_n(t) \mathbf{e}_n,$$

(4.1)

where $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$—that is, $\mathbf{e}_1 = (1,0,0,\ldots,0)$, $\mathbf{e}_2 = (0,1,0,0,\ldots,0)$, $\ldots$, $\mathbf{e}_n = (0,0,0,0,\ldots,1)$—and $x_i: I \to \mathbb{R}$ is a scalar function for each $i = 1, \ldots, n$. You may recall that a function $r$ given by (4.1) is continuous if and only if each of its component functions $x_i$ is continuous for $i = 1, \ldots, n$. Moreover, $r$ is differentiable if and only if each $x_i$ is differentiable, in which case

$$\frac{dr}{dt}(t) = x'_1(t) \mathbf{e}_1 + x'_2(t) \mathbf{e}_2 + \cdots + x'_n(t) \mathbf{e}_n.$$

The parametrized curve $r$ will be called smooth if

- $r$ is differentiable with continuous derivative, and
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• $\frac{dr}{dt}$ is never 0 inside $I$—that is, there is no $t$ on the interior of $I$ for which the component function derivatives $x_i'(t)$ are simultaneously equal to 0.

For our purposes, we may assume that $n = 2$ or $n = 3$. In the former case, $\mathbf{r} : I \subset \mathbb{R} \rightarrow \mathbb{R}^2$ is given by

$$\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j}, \quad t \in I,$$

where $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. In the latter case, $\mathbf{r} : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ is given by

$$\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}, \quad t \in I,$$

where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$.

The independent variable $t$ in (4.1) is called a parameter, and the component functions $x_i$, $i = 1, \ldots, n$, and interval $I$ are essential features of the parametrization $\mathbf{r}$. Two parametrizations are considered the same if and only if the interval $I$ over which the parameter roams is the same for both, and the component functions are the same as well.

The collection of points $\{\mathbf{r}(t) \in \mathbb{R}^n \mid t \in I\}$—the thing we call the curve, or the path traced out by $\mathbf{r}$—is really the range of the parametrization $\mathbf{r}$. It is important to note that a curve can have multiple different parametrizations, as the next example shows.

**Example 4.1.1**

The two vector functions

$$\mathbf{r}_1(t) = t \mathbf{i} + (1 - t) \mathbf{j}, \quad t \in [0, 1],$$

and

$$\mathbf{r}_2(t) = (t^5) \mathbf{i} + (1 - t^5) \mathbf{j}, \quad t \in [0, 1],$$

both trace out the same path—a line segment joining the point $(0, 1)$ in the plane to the point $(1, 0)$. In fact, both involve parameter values in the same interval $I = [0, 1]$. Nevertheless, these are different parametrizations because the component functions of $\mathbf{r}_1$ are not identical to the component functions of $\mathbf{r}_2$, so that a particle tracing out the curves is, in general, in a different location at time $t$ when parametrized by $\mathbf{r}_1$ than it is for $\mathbf{r}_2$ (or, said another way, $\mathbf{r}_1(t) \neq \mathbf{r}_2(t)$, in general). One can use Octave to illustrate this. In the code below, we create a vector $\mathbf{t}$ that partitions the interval $[0, 1]$ with distinct $t$-values equally spaced 0.01 apart. We then plot the points from $\mathbf{r}_1$ corresponding to these distinct $t$-values. The result appears in Figure 4.1. We follow up this command with a plot of the points from $\mathbf{r}_2$ corresponding to these same $t$-values. Whereas the velocity along $\mathbf{r}_1$, given by $\frac{dr_1}{dt} = \mathbf{i} - \mathbf{j}$, is constant, it is clear both in Figure 4.2 and in the formula $\frac{dr_2}{dt} = 5t^4(\mathbf{i} - \mathbf{j})$ that velocity along $\mathbf{r}_2$ changes with $t$. (Recall that it is evenly-spaced parameter values $t$ which are producing the non evenly-spaced points in Figure 4.2.)

```
octave-2.9.13:1> t = 0:.01:1;
octave-2.9.13:2> plot(t, 1-t, 'o')
octave-2.9.13:2> plot(t.^5, 1-t.^5, 'o')
```
4.1 Parametrized Curves and Surfaces

![Figure 4.1: Graph of curve \( \mathbf{r}_1 \)](image)

![Figure 4.2: Graph of curve \( \mathbf{r}_2 \)](image)

As we have noted, the particle that traces out the path according to \( \mathbf{r}_1 \) moves with constant velocity. At \( t = 0 \) it is located at \( \mathbf{j} = (0, 1) \) and at \( t = 1 \) it is at \( \mathbf{i} = (1, 0) \). It is easy to see that, corresponding to a jump of 1 in \( t \)-values, the particle has moved a distance of \( \sqrt{2} \) in the plane. This is borne out in looking at the speed (i.e., the length of the velocity vector) along \( \mathbf{r}_1 \):

\[
\left\| \frac{d\mathbf{r}_1}{dt} \right\| = \| \mathbf{i} - \mathbf{j} \| = \sqrt{1^2 + 1^2} = \sqrt{2}.
\]

There are infinitely many other parametrizations for this same line segment, even sticking to ones that have the same orientation (i.e., ones moving from \((0, 1)\) to \((1, 0)\)). A parametrization that does so with unit constant speed is

\[
\mathbf{r}_3(s) = \left( \frac{s}{\sqrt{2}} \right) \mathbf{i} + \left( 1 - \frac{s}{\sqrt{2}} \right) \mathbf{j}, \quad s \in [0, \sqrt{2}].
\]

Whenever a smooth parametrization \( \mathbf{r} : [0, b] \to \mathbb{R}^n \) satisfies \( \| \mathbf{r}'(s) \| = 1 \) for all \( s \in [0, b] \), we say that \( \mathbf{r} \) is an arc length parametrization of its curve. The parameter \( s \) (the traditional letter to use in such a case) is called the arc length parameter.

**Example 4.1.2**

The vector functions

\[
\mathbf{r}_1(t) = (t) \mathbf{i} + (t^{1/3}) \mathbf{j}, \quad t \in [0, 1],
\]

and

\[
\mathbf{r}_2(t) = (t) \mathbf{i} + (t^{1/3}) \mathbf{j}, \quad t \in [-1, 1],
\]

have the same component functions, but as parametrizations of curves they are different because their domains are different.
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We will sometimes want to deal with curves that naturally break into separate curves. In the next example we find a piecewise smooth parametrization.

**Example 4.1.3**

Find a piecewise smooth parametrization for the union of the line segment $C_1$ from $(0, 0, 1)$ to $(1, 1, 1)$ followed by the line segment $C_2$ from $(1, 1, 1)$ to $(1, 1, 0)$.

To do this, we can find parametrizations for $C_1$ and $C_2$ separately. For $C_1$, take

$$\mathbf{r}_1(t) = (t) \mathbf{i} + (t) \mathbf{j} + \mathbf{k}, \quad t \in [0, 1],$$

and for $C_2$ take

$$\mathbf{r}_2(t) = \mathbf{i} + \mathbf{j} + (2 - t) \mathbf{k}, \quad t \in [1, 2].$$

Both $\mathbf{r}_1$ and $\mathbf{r}_2$ are smooth—in fact, they have constant speeds. We have even gone to the trouble to make them defined over adjoining intervals, with $\mathbf{r}_1(1) = \mathbf{r}_2(1)$, so where the one leaves off, the other one picks up. The full curve $C = C_1 \cup C_2$ is parametrized by the piecewise smooth function $\mathbf{r}$ which equals $\mathbf{r}_1$ in $[0, 1]$ and $\mathbf{r}_2$ in $[1, 2]$.

Finally, we note that, given any smooth parametrized curve $\mathbf{r}: [a, b] \to \mathbb{R}^n$, there exists an arc length parametrization of the same curve having the same orientation. We can use the original parametrization $\mathbf{r}$ to generate this alternate one in the following manner. We let $L(t)$ be the arc length function

$$L(t) := \int_a^t \|\mathbf{r}'(u)\| \, du, \quad t \in [a, b]. \quad (4.2)$$

The Fundamental Theorem of Calculus Part I gives us that $L'(t) = \|\mathbf{r}'(t)\| > 0$ by virtue of $\mathbf{r}$ being smooth. This means $L$ is an increasing function on $[a, b]$, and thus has an inverse. Our arc length parametrization of the curve is then

$$g(s) := \mathbf{r}(L^{-1}(s)), \quad 0 \leq s \leq L(b). \quad (4.3)$$

4.1.2 Surfaces

We see in MATH 162 that surfaces arise naturally as graphs of functions of two variables $z = f(x, y)$. The goal of this section is to extend to surfaces the notion of parametrization we have already for curves, doing so in such a way that our surfaces of MATH 162 fit this scheme.

For a parametrized curve $\mathbf{r}: I \to \mathbb{R}^n$, the domain $I$ (some interval of the real line) gets transformed into a path or trajectory in $\mathbb{R}^n$. While that path may twist and turn, even intersecting itself, it is still essentially a 1-dimensional object; a microscopic, 1-dimensional being residing on this path would be unable to detect its twists and turns, thinking it a “straight-line world”. If we think of a surface as the next-higher-dimensional thing similar to a curve, then we might expect it to be a mapping of some patch of $\mathbb{R}^2$ into $\mathbb{R}^3$—that is,
a function \( r: D \subset \mathbb{R}^2 \to \mathbb{R}^3 \). The next example shows how the kinds of surfaces studied in MATH 162 fit into this framework.

**Example 4.1.4**

Consider the function \( z = x^2 - y^2 \), which you are likely to have studied in MATH 162. The function requires a 2-vector of inputs \((x, y)\). To produce its graph, depicted in Figure 4.3, we plot points \((x, y, x^2 - y^2)\) which correspond to each possible input vector—that is, to each input \((x, y)\) is associated the point \((x, y, x^2 - y^2)\) on the graph, constituting a mapping from \(\mathbb{R}^2\) into \(\mathbb{R}^3\). Just as with a curve, we might think of the inputs \((x, y)\) as parameters specifying where you are on the surface. In this particular example, the parameters are immediately used as the 1\textsuperscript{st} and 2\textsuperscript{nd} components of the output, which is always the easiest thing to do when our surface is the graph of a function of two variables.

Next is an example of a surface that is *not* the graph of a function.

**Example 4.1.5**

As you know, there is no single function of two variables whose graph is the unit sphere (the outer shell of a ball of radius 1). (Think *vertical line test* here; most lines parallel to the coordinate planes which intersect the sphere once do so twice.) However, as you might expect from the study of spherical coordinates, we can map the subset

\[
D = \{ (\phi, \theta) \mid \phi \in [0, \pi], \theta \in [0, 2\pi] \}
\]

of \(\mathbb{R}^2\) onto the unit sphere in \(\mathbb{R}^3\) using the mapping given by

\[
r(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi).
\]

\[
\text{Figure 4.3: Graph of } z = x^2 - y^2
\]

\[
\text{Figure 4.4: Graph of } r(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)
\]
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In Example 4.1.5, the components of the input vector \((\phi, \theta)\) are parameters. Their values are used to determine the three outputs \((x, y, z) = \mathbf{r}(\phi, \theta)\) which are points on the sphere. The parametrization of the sphere given in that example somehow transforms the rectangular patch \(D \subset \mathbb{R}^2\) into a sphere; one can keep track of where everything in \(D\) goes by imagining “lines of latitude and longitude” laid out on our sphere, the former running from 0 (at the “north pole”) to \(\pi\) (at the south), the latter from 0 to \(2\pi\).

This, then, is what we will take a **parametrization of a surface in** \(\mathbb{R}^3\) to be. It is a continuous mapping from some region \(D\) of \(\mathbb{R}^2\) (2-parameter space) into \(\mathbb{R}^3\), that is injective on the interior of \(D\). The outputs of the mapping are points on our surface. That is, it is a mapping

\[
\mathbf{r}: D \subset \mathbb{R}^2 \to \mathbb{R}^3
\]

taking the form

\[
\mathbf{r}(u, v) = f(u, v) \mathbf{i} + g(u, v) \mathbf{j} + h(u, v) \mathbf{k},
\]

for \((u, v) \in D\). (You might stop and think how similar this is, except for the number of parameters, to a parametrization of a curve.) The parametrized surface is called **smooth** if

- each of the partial derivatives

\[
\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial u}, \quad \text{and} \quad \frac{\partial f}{\partial v}
\]

are continuous, and

- the normal vector to the surface at inputs \((u, v) \in D\) given by \(\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v\), where

\[
\mathbf{r}_u := \left( \frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right) = \frac{\partial f}{\partial u} \mathbf{i} + \frac{\partial g}{\partial u} \mathbf{j} + \frac{\partial h}{\partial u} \mathbf{k}, \quad \text{and} \quad \mathbf{r}_v := \left( \frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v} \right),
\]

is never zero on the interior of \(D\). (Note that \(\partial \mathbf{r}/\partial u\) is another way to write \(\mathbf{r}_u\).)

**Example 4.1.6**

Here are several other interesting parametrized surfaces

\[
\mathbf{r}_1(s, t) = (2s - 1) \cos t \mathbf{i} + (2s - 1) \sin t \mathbf{j} + \left( \frac{t}{5} \right) \mathbf{k}, \quad s \in [0, 1], \ t \in \mathbb{R},
\]

\[
\mathbf{r}_2(u, v) = [(2 + \cos u) \cos v] \mathbf{i} + [(2 + \cos u) \sin v] \mathbf{j} + (\sin u) \mathbf{k}, \quad u \in [0, 2\pi], \ v \in [0, 2\pi],
\]

\[
\mathbf{r}_3(s, t) = (\sin^3 s \cos^3 t) \mathbf{i} + (\sin^3 s \sin^3 t) \mathbf{j} + (\cos^3 s) \mathbf{k}, \quad s \in [0, \pi], \ t \in [0, 2\pi].
\]

The surfaces \(\mathbf{r}_1, \mathbf{r}_2\) and \(\mathbf{r}_3\) appear in Figures 4.5, 4.6, and 4.7 respectively.
4.1 Parametrized Curves and Surfaces

Figure 4.5: A helicoid

Figure 4.6: A torus

Figure 4.7: Has name?

Exercises

4.1 TRUE OR FALSE. An arc length parametrization of a curve is always smooth. Explain your answer.

4.2 Consider the vector functions

\[ r_1(t) = (t) \mathbf{i} + (t^2) \mathbf{j}, \quad t \in [-1, 1], \]

and

\[ r_2(t) = (1 - 2t^2) \mathbf{i} + (1 - 2t^2)^2 \mathbf{j}, \quad t \in [-1, 1]. \]

a) Both \( r_1 \) and \( r_2 \) produce the same curve. Sketch it. If you wanted a function \( y = f(x) \), \( a \leq x \leq b \) whose graph was this curve, what function \( f \) and endpoints \( a, b \) to its domain would do the job?

b) Recall that the length of a parametrized curve \( r : [a, b] \rightarrow \mathbb{R}^n \) of the form (4.1) is given by

\[ L := \int_a^b \sqrt{\left( \frac{dx_1}{dt} \right)^2 + \left( \frac{dx_2}{dt} \right)^2 + \cdots + \left( \frac{dx_n}{dt} \right)^2} \, dt = \int_a^b \|dr/dt\| \, dt. \]

For all but a few specialized cases, it is difficult to antidifferentiate \( \|dr/dt\| \), making it more practical to approximate the arc length using a numerical integration method like Simpson’s rule than to evaluate it via the Fundamental Theorem of Calculus, Part II. (You might quickly peruse Section 7.6 of University Calculus to remind yourself of relevant details involving Simpson’s Rule.) Use the applet at http://www.math.ucla.edu/~ronmiech/Java_Applets/Riemann/index.html to find approximate arc lengths associated with \( r_1 \) and \( r_2 \). They will not be equal. Explain why this is so.
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c) Show that \( r_1 \) is a smooth parametrization, but that \( r_2 \) is not.

4.3 Suppose \( r(s), s \in [0,b] \), is an arc length parametrization of some curve, and that \( r \) has the form \((4.1)\). Show that, for each \( s \in [0,b] \), the value of the arc length function \( L \) in \((4.2)\) associated with \( r \) is \( s \).

4.4 Find a smooth or piecewise smooth parametrization for the given curves in \( \mathbb{R}^2 \).

a) The unit circle (i.e., the circle centered at 0 with radius 1), oriented clockwise.

b) The circle centered at \((h,k)\) with radius \( r \), oriented counterclockwise.

c) The rectangle with corners at \((1,3)\), \((3,3)\), \((3,8)\), and \((1,8)\), traversed counterclockwise.

d) The triangle with corners at \((-2,-1)\), \((5,-1)\), and \((5,3)\), oriented counterclockwise.

4.5 Professor Will Dickinson of the mathematics department at Grand Valley State University oversees an annual “Parametric Curves Contest” at the school. Many (perhaps all?) entries are viewable from the website http://faculty.gvsu.edu/dickinsw/ParametricEntries.html. Visit this website and view some of the entries. Select one and determine parametrizations which produce a reasonable facsimile of the picture. Submit the year, the name of the picture you are mimicking, and your functions from \( \mathbb{R} \) into \( \mathbb{R}^2 \) which result in the picture.

4.6 Consider the plane in \( \mathbb{R}^3 \) passing through the three points \((0,0,1)\), \((0,1,0)\), and \((1,0,0)\).

a) Find a normal vector to this plane.

b) Use the normal vector from part a) to determine an equation in \( x, y, z \) for the plane.

c) Find a parametrization of the plane from part b).

4.7 Consider the curve traced out by

\[
(cos \ t) \mathbf{i} + (sin \ t) \mathbf{j} + (t) \mathbf{k}, \quad t \in [0,2\pi].
\]

a) Sketch this curve in \( \mathbb{R}^3 \).

b) Show that this is a smooth parametrization.

c) Use equation \((4.3)\) to find an arc length parametrization \( g \) of this curve.

4.8 For each of the three parametrized surfaces \( r_1, r_2, \) and \( r_3 \) of Example 4.1.6, sketch the region \( D \) of \( \mathbb{R}^2 \) being mapped to the surface.
4.2 Line Integrals (Outline)

The Main idea: There is some quantity (function) \( f = f(x) \) that changes as you move from one location \( x \) to another in \( \mathbb{R}^n \) (\( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)). We wish to sum the values (i.e., take an integral) of \( f \) along some designated curve \( C \).

When \( C \) is a curve in \( \mathbb{R}^2 \), the resulting integral is denoted
\[
\int_C f(x, y) \, ds ,
\]
and is called the line integral of \( f \) over \( C \). (There is the obvious modification if \( C \) is a curve in \( \mathbb{R}^3 \).) To compute the integral in (4.4) we

1. must have (or produce) a parametrization \( r(t) = g(t) \mathbf{i} + h(t) \mathbf{j} \), \( t \in [a, b] \), of the curve \( C \), and

2. use the relationship
\[
\int_C f(x, y) \, ds = \int_a^b f(g(t), h(t)) \|r'(t)\| \, dt .
\]

Line integrals are additive, so that if a curve \( C \) breaks naturally into finitely many parts (sub-curves) \( C_1, \ldots, C_k \), then
\[
\int_C f(x, y) \, ds = \int_{C_1} f(x, y) \, ds + \int_{C_2} f(x, y) \, ds + \cdots + \int_{C_k} f(x, y) \, ds .
\]

4.3 Vector Fields, Work, Circulation, Flux (Outline)

A vector field is a function \( F: D \subset \mathbb{R}^n \to \mathbb{R}^n \) (\( n = 2 \) or \( n = 3 \)). In the case \( n = 3 \),
\[
F(x, y, z) = M(x, y, z) \mathbf{i} + N(x, y, z) \mathbf{j} + P(x, y, z) \mathbf{k} , \quad (x, y, z) \in D .
\]

Examples of vector fields:
- velocity, magnetic, gravitational, and electric fields.
- for a differentiable function \( f: \mathbb{R}^n \to \mathbb{R} \) the gradient vector \( \nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n) \) is a vector field.

The line integral of a continuous vector field \( F \) over a smooth parametrized curve \( C: r(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \), \( t \in [a, b] \), is written in many forms, some of which are
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \left\| \frac{d\mathbf{r}}{dt} \right\| \, dt = \int_a^b \mathbf{F} \cdot \mathbf{T} \, ds
\]
\[
= \int_a^b \left( M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) \, dt
\]
\[
= \int_a^b M \, dx + N \, dy + P \, dz ,
\]
(4.6)
where $\mathbf{T}$ is the unit tangent vector, and $ds = \|d\mathbf{r}/dt\| \, dt$. The terms involving $P$ are absent when $\mathbf{F}$ and $\mathbf{r}$ are a vector field and curve in the plane ($\mathbb{R}^2$). When $\mathbf{r}(a) = \mathbf{r}(b)$ (i.e., when the curve $C$ ends at its starting point), then $C$ is called a **closed curve**. In this instance, the first form of the line integral (4.6) is often written as

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$ 

Interpretation of the line integral (4.6) (in all its forms), depends on context.

- For a force field $\mathbf{F}$, (4.6) gives the **work** done moving an object over the curve $\mathbf{r}$.
- If $\mathbf{F}$ is a velocity field, (4.6) is a **flow integral**. When $\mathbf{r}$ is a closed curve, this flow integral is called the **circulation** around $C$.

### 4.4 Path Independence, Potential Functions, and Conservative Fields (Outline)

For what follows, assume $D \subset \mathbb{R}^n$ is an open set. (See p. 704 and p. 707 of *University Calculus* for clarification.)

**Definition 4.4.1.** A vector field $\mathbf{F}: D \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be **conservative on** $D$ if, given any parametrized curve $C$ in $D$ parametrized by $\mathbf{r}$, the value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of the curve $C$.

It is a fact that $\mathbf{F}$ is conservative if and only if either of the following conditions holds:

- All line integrals are **path independent**. That is, given two curves $C_1, C_2$ in $D$ parametrized by $\mathbf{r}_1$ and $\mathbf{r}_2$, and having the same starting and ending points, we have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2.$$  

- Given any closed curve $C$ in $D$, the line integral of $\mathbf{F}$ over $C$ is zero: $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

If we know more about $\mathbf{F}$ and the region $D$, then we have other tests for conservativeness.

**Theorem 4.4.2.** Suppose $\mathbf{F} = Mi + Nj +Pk$ is a continuous vector field on an open, connected set $D \subset \mathbb{R}^3$ (i.e., each of $M$, $N$, and $P$ are continuous throughout $D$). Then $\mathbf{F}$ is conservative on $D$ if and only if there is a scalar-valued function $f: D \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$. If such a function $f$ exists, then for a curve $C$ in $D$ with endpoints $\mathbf{a}$ and $\mathbf{b}$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$
4.5 Green’s Theorem in the Plane (Outline)

If a vector field \( \mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \) on \( D \subset \mathbb{R}^3 \) satisfies \( \mathbf{F} = \nabla f \) (the gradient of \( f \)) for some scalar-valued function \( f \) on \( D \), then \( f \) is called a **potential function** for \( \mathbf{F} \). When such an \( f \) exists, the **differential form**

\[
M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = df
\]

is said to be **exact**.

**Theorem 4.4.3.** Suppose \( \mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k} \) is a vector field on a **connected** and **simply connected** open region \( D \subset \mathbb{R}^n \), and that the components of \( \mathbf{F} \) have continuous partial derivatives throughout \( D \). Then \( \mathbf{F} \) is conservative on \( D \) if and only if

\[
\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},
\]

or, equivalently,

\[
\text{curl} \mathbf{F} = \nabla \times \mathbf{F} := \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
M & N & P
\end{vmatrix}
\]

\[
= \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left( \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} = \mathbf{0} .
\]

Both the last two theorems hold for vector fields on similar regions \( D \subset \mathbb{R}^2 \); we just drop any statements made about the (non-existent) component function \( P \) in that case.

### 4.5 Green’s Theorem in the Plane (Outline)

#### 4.5.1 Interpretation of the curl and divergence

**The Curl.** In a **velocity field** \( \mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \), one can imagine placing an infinitesimally small **paddle wheel** (see Figures 14.27, 14.54 of the text) at a point \( Q = (x, y, z) \). Depending upon its axis of rotation, the motion in the velocity field may cause the paddle wheel to spin or not. The curl vector (4.7) gives an axis of rotation that maximizes this **circulation density**. Specifically, the rotation of the velocity field (in the plane normal to the curl vector) and the curl vector follow a right-hand rule, and the length of the curl vector quantifies the rate of the fluid’s rotation about this axis.

While there is no corresponding thing called a **curl** of a vector field \( \mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} \) in \( \mathbb{R}^2 \), the expression \( (\partial N/\partial x - \partial M/\partial y) \) is a **signed scalar** quantifying rotation of the field. If \( (\partial N/\partial x - \partial M/\partial y) \) is positive at a point \( Q \) in the plane, then there is **counterclockwise rotation** at \( Q \); if it is negative, the rotation is clockwise.

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The Divergence. For a 3-dimensional vector field \( \mathbf{F}(x, y, z) = M \mathbf{i} + N \mathbf{j} + P \mathbf{k} \), the divergence (or flux density) of \( \mathbf{F} \) is given by

\[
\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} := \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.
\]

(For 2-dimensional \( \mathbf{F}(x, y) = M \mathbf{i} + N \mathbf{j} \), we have \( \text{div } \mathbf{F} = \partial M/\partial x + \partial N/\partial y \).) Note that \( \text{div } \mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R} \) assigns to each point in \( Q \in D \) a scalar value. When positive, it indicates \( Q \) is a source; if \( \mathbf{F} \) is the velocity field for a gas, then at a source point gas is expanding. When \( \text{div } \mathbf{F} \) is negative, \( Q \) is a sink.

4.5.2 Green’s Theorem in the plane

Some line integrals involving piecewise smooth curves are tedious to calculate. A parametrization around a rectangular path has 4 pieces and, if the line integral is of a nonconservative vector field, then the integral over each side must be computed separately. In light of this, Green’s Theorem in the plane has something to say if you are

- integrating around a piecewise smooth, simple closed curve \( C \),
- integrating a vector field \( \mathbf{F} = M \mathbf{i} + N \mathbf{j} \) in \( \mathbb{R}^2 \), whose components \( M \) and \( N \) have continuous first partial derivatives in an open region \( D \subset \mathbb{R}^2 \) containing \( C \) and the region it encloses, and
- calculating the circulation of \( \mathbf{F} \) around \( C \) or the flux of \( \mathbf{F} \) across \( C \).

Under these conditions, we have

1. the counterclockwise circulation of \( \mathbf{F} \) around \( C \) equals the double integral of circulation density over the region \( R \) enclosed by curve \( C \); that is,

\[
\oint_C \mathbf{F} \cdot \mathbf{T} \, ds := \oint_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy.
\]

(4.9)

2. The outward flux of \( \mathbf{F} \) across \( C \) equals the double integral of the flux density over the region \( R \) enclosed by curve \( C \); that is

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds := \oint_C M \, dy - N \, dx = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dx \, dy.
\]

(4.10)

4.6 Surface Integrals and Area (Outline)

Given a smooth parametrization of a surface

\[
\mathbf{r}(u, v) = f(u, v) \mathbf{i} + g(u, v) \mathbf{j} + h(u, v) \mathbf{k}, \quad (u, v) \in D \subset \mathbb{R}^2,
\]

(4.11)
we may approximate the area of the surface patch corresponding to a rectangular region in \((u, v)\)-parameter space of area \(\Delta u \Delta v\) by

\[
\|r_u\| \|r_v\| \sin \theta \, \Delta u \Delta v = \|r_u \times r_v\| \Delta u \Delta v .
\]

In the limit, as the side lengths \(\Delta u, \Delta v \to 0\), we get

- a smooth surface \(S\) parametrized by (4.11) has surface area

\[
\text{Area} = \iint_S d\sigma = \iint_D \|r_u \times r_v\| \, dA . \tag{4.12}
\]

- if there is some scalar quantity \(G(x, y, z)\) that varies depending upon location in \(\mathbb{R}^3\), and we wish to add up values of \(G\) over a particular surface \(S\) with smooth parametrization (4.11), the value of the surface integral of \(G\) over \(S\) is

\[
\iint_S G(x, y, z) \, d\sigma = \iint_D G(f(u, v), g(u, v), h(u, v)) \|r_u \times r_v\| \, dA . \tag{4.13}
\]

Some surfaces \(S\), such as the faces of a cube, are piecewise smooth rather than smooth. In this instance, if \(S_1, \ldots, S_6\) are smooth parametrizations of the 6 faces, then

\[
\iint_S G(x, y, z) \, d\sigma = \sum_{j=1}^6 \iint_{S_j} G(x, y, z) \, d\sigma .
\]
4 Vector Calculus

4.7 Flux across a Surface (Outline)

Equation (3) of p. 863 in University Calculus deals with planar flow across a curve. The flux-divergence form (equation (4.10)) of Green’s theorem provides an alternate method for calculating such an integral under special circumstances. (See the last section.) But since our typical experience of flow is 3-dimensional, not planar, we need analogs for these concepts in 3 dimensions.

In 3D-space, we talk about flow (flux) of a vector field \( \mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \) across an oriented surface—that is, a surface upon which it is possible to define a unit normal vector field \( \mathbf{n} \) that varies continuously with position. (That non-orientable surfaces exist is illustrated by the Möbius band of Figure 14.45, p. 899.) At any point \( Q \) on the surface, \( \mathbf{n} \) is called the positive direction at \( Q \). Given a smooth oriented surface \( S \) with parametrization \( \mathbf{r}: D \subset \mathbb{R}^2 \to \mathbb{R}^3 \), we define the flux integral of \( \mathbf{F} \) over \( S \) to be

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma := \iint_D \mathbf{F} \cdot \left( \frac{\mathbf{r}_u \times \mathbf{r}_v}{\lVert \mathbf{r}_u \times \mathbf{r}_v \rVert} \right) \lVert \mathbf{r}_u \times \mathbf{r}_v \rVert \, dA
\]

\[
= \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv \tag{4.14}
\]

It may be useful to draw parallels between concepts we have investigated, as is done in the following table.

<table>
<thead>
<tr>
<th>Lines (curves)</th>
<th>Surfaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parametrization</td>
<td>( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} )</td>
</tr>
<tr>
<td>Parametrization</td>
<td>( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} )</td>
</tr>
<tr>
<td>Basic unit</td>
<td>( ds = \lVert d\mathbf{r}/dt \rVert , dt )</td>
</tr>
<tr>
<td>Integrals of scalar functions</td>
<td>( \int_C f(x, y) , ds ) (planar)</td>
</tr>
<tr>
<td>Integrals of scalar functions</td>
<td>( \int_C f(x, y, z) , ds ) (spatial)</td>
</tr>
<tr>
<td>Flux of vector field ( \mathbf{F} )</td>
<td>( \int_C \mathbf{F}(x, y) \cdot \mathbf{n} , ds ) (planar)</td>
</tr>
<tr>
<td>Flux of vector field ( \mathbf{F} )</td>
<td>( \int_C \mathbf{F}(x, y, z) \cdot \mathbf{n} , ds ) (spatial)</td>
</tr>
</tbody>
</table>
4.8 Stokes’ Theorem (Outline)

4.8.1 The result, generalizing Green’s Theorem

For the theorem, we suppose that

- $S$ is a piecewise smooth oriented surface (i.e., the finite
  union of smooth surfaces), with piecewise smooth edge
  (or boundary) curve $C$, and

- $\mathbf{F} = Mi + Nj +Pk$ is a vector field whose compo-
  nent functions $M$, $N$, $P$ have continuous first partial
  derivatives in an open region of $\mathbb{R}^3$ containing $S$.

**Theorem 4.8.1 (Stokes’ Theorem).** Under the conditions stated above, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma,$$

(4.15)

where $C$ is oriented so that if a person were walking along $C$ standing with head facing the
positive direction, the surface $S$ stays on the left-hand side. (See the figure at right above,
along with more complicated ones like Figure 14.58, p. 911.)

Remarks:

1. If two different piecewise smooth oriented surfaces $S_1$ and $S_2$ both lie inside the open
region of $\mathbb{R}^3$ where $M$, $N$, and $P$ have continuous first partial derivatives, and if they
are oriented in the same way, then the above theorem implies

$$\iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

2. The circulation-curl form of Green’s theorem may be seen as a special case of Stokes’
theorem.

4.8.2 The fundamental theorems and their relationships

In the calculus sequence we have seen various fundamental theorems which allow one to
evaluate an integral over an “interior” region by evaluating a related function or integral
on the boundary. In the table below, the main results are amassed together without all the
assumptions/hypotheses sufficient to make them true. The horizontal lines serve to group
together “like” theorems.
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<table>
<thead>
<tr>
<th>Theorem Name</th>
<th>Main Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>FT of Calculus:</td>
<td>If $F'(x) = f(x)$, then $\int_a^b f(x),dx = F(b) - F(a)$.</td>
</tr>
<tr>
<td>FT of Line Integrals:</td>
<td>If $\mathbf{F} = \nabla f$, and the curve $C$ has endpoints $A$ and $B$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$.</td>
</tr>
<tr>
<td>Green’s Theorem:</td>
<td>$\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \oint_C \mathbf{F} \cdot d\mathbf{r}$ (circulation-curl form)</td>
</tr>
<tr>
<td>Stokes’ Theorem:</td>
<td>$\iint_S \mathbf{\nabla} \times \mathbf{F} \cdot \mathbf{n} ,d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$, where $C$ is the edge curve of $S$</td>
</tr>
<tr>
<td>Green’s Theorem:</td>
<td>$\iint_R \mathbf{\nabla} \cdot \mathbf{F} ,dA = \oint_C \mathbf{F} \cdot \mathbf{n} ,ds$ (flux-divergence form)</td>
</tr>
<tr>
<td>Divergence Theorem:</td>
<td>$\iiint_R \mathbf{\nabla} \cdot \mathbf{F} ,dV = \iint_S \mathbf{F} \cdot \mathbf{n} ,d\sigma$ (See the next section.)</td>
</tr>
</tbody>
</table>

### 4.9 The Divergence Theorem (Outline)

Speaking loosely, the term **closed surface** in the following theorem refers to a surface $S$ that encloses a solid region of $\mathbb{R}^3$.

**Theorem 4.9.1** (Divergence Theorem). Suppose $S$ is a piecewise smooth oriented closed surface with *outward-pointing* normal $\mathbf{n}$, and $\mathbf{F}$ is a vector field whose components have continuous first partial derivatives in an open connected region of $\mathbb{R}^3$ containing both $S$ and the region $R$ enclosed by $S$. Then

$$\iiint_S \mathbf{F} \cdot \mathbf{n} \,d\sigma = \iiint_R \mathbf{\nabla} \cdot \mathbf{F} \,dV . \tag{4.16}$$

As noted above, the Divergence Theorem is a fundamental theorem of calculus, generalizing the Fundamental Theorem studied in MATH 161. There are several corollaries to this theorem, namely

$$\iiint_S f \nabla g \cdot \mathbf{n} \,d\sigma = \iiint_R \left( f \nabla^2 g + \mathbf{\nabla} f \cdot \nabla g \right) dV , \tag{4.17}$$

and

$$\iiint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} \,d\sigma = \iiint_D (f \nabla^2 g - g \nabla^2 f) dV \tag{4.18}$$
4.9 The Divergence Theorem (Outline)

(called Green’s 1st and 2nd identities respectively), which follow from the Divergence Theorem under appropriate assumptions, and generalize the notion of integration by parts. See Exercises 29 and 30 from Section 14.8 of University Calculus.