Not a Simple Renaming of an Arbitrary Constant (NSRAC)

It is common practice while solving differential equations to treat arbitary constants with a broad stroke. For instance, when integrating both sides of a separable DE

$$h(y)y' \;=\; g(x),$$

we should get arbitary constants from both integrals:

$$H(y) + C_1 \;=\; G(x) + C_2.$$

One may do a little algebra on the previous equation to arrive at

$$H(y) \;=\; G(x) + C_2 - C_1,$$

and then argue that the difference $C_2 - C_1$ of two arbitrary constants is just another arbitrary constant. This happens so often that we come to leave out the intermediary steps:

$$h(y)y' \;=\; g(x) \qquad \rightarrow \qquad H(y) \;=\; G(x) + C.$$

In other situations, we often rename constants. We might have solved the DE to get

$$\frac{y}{2} = e^{x/5} + C,$$

and multiply through by 2 to obtain

$$y = 2e^{x/5} + 2C.$$

Once again, twice an arbitrary constant can be thought of as a new arbitrary constant. Often we are so flippant about it, we reuse the same letter, calling both the original arbitrary constant, and the new one (which is twice as large), by the name "C"

$$\frac{y}{2} = e^{x/5} + C \qquad \Rightarrow \qquad y = 2e^{x/5} + C.$$

But this sort of flippancy comes with rules. In particular, throughout this process we must *stay within the same family of functions*.

An example of an error: Students will sometimes have

$$y^3 \;=\; x + C,$$

and wanting to solve for y, they get

$$y \;=\; \sqrt[3]{x} + C,$$

when the correct answer would be

$$y \;=\; \sqrt[3]{x + C}.$$

The error here is in asserting that

$$\sqrt[3]{x+C} \qquad \text{and} \qquad C + \sqrt[3]{x}$$

are really the same family of functions, but for a simple renaming of the arbitrary constant. But this is not so. Altering the value of C in $\sqrt[3]{x+C}$ shifts the graph of $y = \sqrt[3]{x}$ right or left, while altering the value of C in $C + \sqrt[3]{x}$ shifts the graph up or down.

Another example: Starting from

$$-\frac{1}{y} = -C + \cos x \qquad \text{and arriving at} \qquad y = C - \frac{1}{\cos x},$$

which is a different family of functions altogether than the correct one

$$y = \frac{1}{C - \cos x}.$$