MATH 231A
Answers to class activity, April 2, 2003

1. We have an existence and uniqueness theorem for nth-order linear differential equations of the type
$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y \;=\; g(t),$$
(a form which the given DE fits), which says that, given a full set of initial conditions
$$y(t_0) = y_0, \;\; y'(t_0) = y_0', \;\; \ldots, \;\; y^{(n-1)}(t_0) = y_0^{n-1},$$
a unique solution exists to the IVP on the largest open interval I containing t_0 on which all of the functions $p_1, \ldots, p_n$ and g are continuous. As our coefficient functions are all constant, this interval will be $(-\infty, \infty)$, no matter what $t_0, y_0, \ldots, y_0^{n-1}$ are.

2. The problem has characteristic equation
$$\lambda^4 + 4\lambda^3 + 13\lambda^2 + 36\lambda + 36 = 0.$$
The graph of $f(\lambda) := \lambda^4 + 4\lambda^3 + 13\lambda^2 + 36\lambda + 36$ reveals that $\lambda = -2$ is an even-multipled root of the characteristic equation (i.e., is a root of order 2 or 4). Taking it to be at least a double root, we know $(\lambda+2)^2 = \lambda^2 + 4\lambda + 4$ is a factor of f. Using long division, we find the other factor to be $\lambda^2 + 9$, which yields to more (nonreal) roots of f, namely $\lambda = \pm 3i$. Thus, we could say that the general solution of the DE is
$$y(t) \;=\; c_1e^{-2t} + c_2te^{-2t} + c_3e^{3it} + c_4e^{-3it}.$$
But, since this solution has nonreal references, we prefer the general solution
$$y(t) \;=\; k_1e^{-2t} + k_2te^{-2t} + k_3\cos(3t) + k_4\sin(3t).$$

3. We convert the DE to a 1st-order system using the substitutions
$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad x_4 = y'''.$$
These substitutions immediately yield three 1st-order DEs:
$$x_1' = x_2, \quad x_2' = x_3, \quad \text{and} \quad x_3' = x_4.$$
The 4th 1st-order DE is the original 4th-order DE itself, now written in the new variables — that is,
$$y'''' = -36y - 36y' - 13y'' - 4y''' \quad \text{becomes} \quad x_4' = -36x_1 - 36x_2 - 13x_3 - 4x_4.$$
Written as a matrix DE, we have
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & -36 & -13 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

The structure seen in the coefficient matrix — a diagonal of 1's appearing just above the main diagonal and all other entries (except in the last row) being 0 — is the same whenever an nth-order linear DE is converted to a system. In theory, Chapter 7 methods will allow us to handle any 1st-order matrix DE (at least, any homogeneous one). So, in theory (if not necessarily in practice) any linear homogeneous nth-order DE may be solved using Ch. 7 methods. The reverse is not true, as one may deduce from the realization that there are a lot 1st-order matrix DEs one could write down which do not have the structure observed above. Without that structure, it would be impossible to do a conversion the other way in order to use Chapter 3 methods.

4. To make it easier to follow the computation of $\det(A - \lambda I)$ below, all expansions have been done along first rows.

$$\begin{aligned}
\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ -36 & -36 & -13 & -4-\lambda \end{vmatrix}
&= (-\lambda)\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -36 & -13 & -4-\lambda \end{vmatrix} - (1)\begin{vmatrix} 0 & 1 & 0 \\ 0 & -\lambda & 1 \\ -36 & -13 & -4-\lambda \end{vmatrix} \\
&= (-\lambda)\left[(-\lambda)\begin{vmatrix} -\lambda & 1 \\ -13 & -4-\lambda \end{vmatrix} - (1)\begin{vmatrix} 0 & 1 \\ -36 & -4-\lambda \end{vmatrix}\right] \\
&\quad + \begin{vmatrix} 0 & 1 \\ -36 & -4-\lambda \end{vmatrix} \\
&= \lambda^2[\lambda(4+\lambda)+13] + 36\lambda + 36 \\
&= \lambda^4 + 4\lambda^3 + 13\lambda^3 + 36\lambda + 36.
\end{aligned}$$

This (as perhaps is no surprise) is the same polynomial that we had in the characteristic equation in part 2.

5. For $\lambda = -2$ we find the associated eigenvector(s) by solving the matrix equation $(A+2I)\xi = \mathbf{0}$. Using Gaussian elimination:

$$\left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ -36 & -36 & -13 & -2 & 0 \end{array}\right)
\begin{array}{c} 18\mathbf{r}_1 + \mathbf{r}_4 \to \mathbf{r}_4 \\ \sim \end{array}
\left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & -18 & -13 & -2 & 0 \end{array}\right)$$

$$\begin{array}{c} 9\mathbf{r}_2 + \mathbf{r}_4 \to \mathbf{r}_4 \\ \sim \end{array}
\left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -4 & -2 & 0 \end{array}\right)$$

$$\begin{array}{c} 2\mathbf{r}_3 + \mathbf{r}_4 \to \mathbf{r}_4 \\ \sim \end{array}
\left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We could (and often do) make all of the pivots be 1 (here each is 2), but it is not necessary. Since there is just one missing pivot, there is only one free variable. This means that, despite the fact that $\lambda = -2$ is an eigenvalue of multiplicity 2, it will have just a line of associated eigenvectors — i.e., if we find one (nonzero) eigenvector associated with $\lambda = -2$, all others will be scalar multiples of that one; **there are not two LI eigenvectors associated with** $\lambda = -2$. Writing the eigenvector $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ and taking ξ_4 to be free (i.e., any value of ξ_4 is allowed), we can set it to be any convenient number and compute the other components ξ_3, ξ_2, and ξ_1. I have taken $\xi_4 = -8$, and then solved the three equations below (on the left) from bottom to top:

$$\begin{aligned} 2\xi_1 + \xi_2 &= 0 \\ 2\xi_2 + \xi_3 &= 0 \\ 2\xi_3 + \xi_4 &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \xi_1 &= 1 \\ \xi_2 &= -2 \\ \xi_3 &= 4. \end{aligned}$$

We have the sense that, since $\lambda = -2$ had multiplicity 2, it should produce two linearly independent solutions of the matrix DE. Thus far, it has only produced one, namely

$$\mathbf{x}^{(1)}(t) \quad = \quad e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \\ -8 \end{pmatrix} \quad = \quad \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \\ -8e^{-2t} \end{pmatrix}.$$

Turning now to the eigenvalue $\lambda = 3i$, we again perform Gaussian elimination:

$$\left(\begin{array}{cccc|c} -3i & 1 & 0 & 0 & 0 \\ 0 & -3i & 1 & 0 & 0 \\ 0 & 0 & -3i & 1 & 0 \\ -36 & -36 & -13 & -4-3i & 0 \end{array}\right) \quad \begin{array}{c} i\mathbf{r}_1 \to \mathbf{r}_1 \\ \sim \\ i\mathbf{r}_2 \to \mathbf{r}_2 \\ i\mathbf{r}_3 \to \mathbf{r}_3 \end{array} \quad \left(\begin{array}{cccc|c} 3 & i & 0 & 0 & 0 \\ 0 & 3 & i & 0 & 0 \\ 0 & 0 & 3 & i & 0 \\ -36 & -36 & -13 & -4-3i & 0 \end{array}\right)$$

$$\begin{array}{c} 12\mathbf{r}_1 + \mathbf{r}_4 \to \mathbf{r}_4 \\ \sim \end{array} \quad \left(\begin{array}{cccc|c} 3 & i & 0 & 0 & 0 \\ 0 & 3 & i & 0 & 0 \\ 0 & 0 & 3 & i & 0 \\ 0 & -36+12i & -13 & -4-3i & 0 \end{array}\right)$$

$$\begin{array}{c} (12-4i)\mathbf{r}_2 + \mathbf{r}_4 \to \mathbf{r}_4 \\ \sim \end{array} \quad \left(\begin{array}{cccc|c} 3 & i & 0 & 0 & 0 \\ 0 & 3 & i & 0 & 0 \\ 0 & 0 & 3 & i & 0 \\ 0 & 0 & -9+12i & -4-3i & 0 \end{array}\right)$$

$$\begin{array}{c} (3-4i)\mathbf{r}_3 + \mathbf{r}_4 \to \mathbf{r}_4 \\ \sim \end{array} \quad \left(\begin{array}{cccc|c} 3 & i & 0 & 0 & 0 \\ 0 & 3 & i & 0 & 0 \\ 0 & 0 & 3 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Again, ξ_4 is free. This time a convenient value seems to be $\xi_4 = -27i$, which then means $\xi_3 = -9$, $\xi_2 = 3i$ and $\xi_1 = 1$ respectively. So, $\lambda = 3i$ has associated eigenvector

$$\xi \;=\; \begin{pmatrix} 1 \\ 3i \\ -9 \\ -27i \end{pmatrix} \;=\; \begin{pmatrix} 1 \\ 0 \\ -9 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 3 \\ 0 \\ -27 \end{pmatrix} \;=\; \mathbf{u} + i\mathbf{v}.$$

Without any more work, we know that $\lambda = -3i$ has associated eigenvector $\mathbf{u} - i\mathbf{v}$. We could take $e^{3it}(\mathbf{u} + i\mathbf{v})$ and $e^{-3it}(\mathbf{u} - i\mathbf{v})$ to be two more L.I. solutions (they indeed are), but we prefer to take the usual linear combinations which yield real solutions:

$$
\begin{aligned}
\mathbf{x}^{(2)}(t) &= \frac{1}{2}\left[e^{3it}(\mathbf{u}+i\mathbf{v}) + e^{-3it}(\mathbf{u}-i\mathbf{v})\right] \\
&= \cos(3t)\begin{pmatrix} 1 \\ 0 \\ -9 \\ 0 \end{pmatrix} - \sin(3t)\begin{pmatrix} 0 \\ 3 \\ 0 \\ -27 \end{pmatrix} = \begin{pmatrix} \cos(3t) \\ -3\sin(3t) \\ -9\cos(3t) \\ 27\sin(3t) \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{x}^{(3)}(t) &= \frac{1}{2i}\left[e^{3it}(\mathbf{u}+i\mathbf{v}) - e^{-3it}(\mathbf{u}-i\mathbf{v})\right] \\
&= \cos(3t)\begin{pmatrix} 0 \\ 3 \\ 0 \\ -27 \end{pmatrix} + \sin(3t)\begin{pmatrix} 1 \\ 0 \\ -9 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin(3t) \\ 3\cos(3t) \\ -9\sin(3t) \\ -27\cos(3t) \end{pmatrix}.
\end{aligned}
$$

We have three L.I. solutions. Knowing that four are needed for a fundamental set of solutions, we do not (yet) have a fundamental set.

6. First, to compare solutions obtained via the two methods, we have to recognize that the four components of our solutions in part 5 represent y, y', y'' and y''' respectively. If we take our solution from part 2 and differentiate, we get

$$
\begin{aligned}
y(t) &= k_1e^{-2t} + k_2te^{-2t} + k_3\cos(3t) + k_4\sin(3t) \\
y'(t) &= (k_2 - 2k_1)e^{-2t} - 2k_2te^{-2t} - 3k_3\sin(3t) + 3k_4\cos(3t) \\
y''(t) &= (4k_1 - 4k_2)e^{-2t} + 4k_2te^{-2t} - 9k_3\cos(3t) - 9k_4\sin(3t) \\
y'''(t) &= (12k_2 - 8k_1)e^{-2t} - 8k_2te^{-2t} + 27k_3\sin(3t) - 27k_4\cos(3t),
\end{aligned}
$$

or, arranging these derivatives in vector form,

$$
\begin{aligned}
\begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix} &= \begin{pmatrix} k_1e^{-2t} + k_2te^{-2t} + k_3\cos(3t) + k_4\sin(3t) \\ (k_2 - 2k_1)e^{-2t} - 2k_2te^{-2t} - 3k_3\sin(3t) + 3k_4\cos(3t) \\ (4k_1 - 4k_2)e^{-2t} + 4k_2te^{-2t} - 9k_3\cos(3t) - 9k_4\sin(3t) \\ (12k_2 - 8k_1)e^{-2t} - 8k_2te^{-2t} + 27k_3\sin(3t) - 27k_4\cos(3t) \end{pmatrix} \\
&= \begin{pmatrix} k_1e^{-2t} + k_2te^{-2t} \\ (k_2 - 2k_1)e^{-2t} - 2k_2te^{-2t} \\ (4k_1 - 4k_2)e^{-2t} + 4k_2te^{-2t} \\ (12k_2 - 8k_1)e^{-2t} - 8k_2te^{-2t} \end{pmatrix} + k_3\begin{pmatrix} \cos(3t) \\ -3\sin(3t) \\ -9\cos(3t) \\ 27\sin(3t) \end{pmatrix} + k_4\begin{pmatrix} \sin(3t) \\ 3\cos(3t) \\ -9\sin(3t) \\ -27\cos(3t) \end{pmatrix} \\
&= \begin{pmatrix} k_1e^{-2t} + k_2te^{-2t} \\ (k_2 - 2k_1)e^{-2t} - 2k_2te^{-2t} \\ (4k_1 - 4k_2)e^{-2t} + 4k_2te^{-2t} \\ (12k_2 - 8k_1)e^{-2t} - 8k_2te^{-2t} \end{pmatrix} + k_3\mathbf{x}^{(2)}(t) + k_4\mathbf{x}^{(3)}(t)
\end{aligned}
$$

$$= k_1 \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \\ -8e^{-2t} \end{pmatrix} + k_2 \begin{pmatrix} te^{-2t} \\ e^{-2t} - 2te^{-2t} \\ -4e^{-2t} + 4te^{-2t} \\ 12e^{-2t} - 8te^{-2t} \end{pmatrix} + k_3 \mathbf{x}^{(2)}(t) + k_4 \mathbf{x}^{(3)}(t)$$

$$= k_1 \mathbf{x}^{(1)}(t) + k_2 \begin{pmatrix} te^{-2t} \\ e^{-2t} - 2te^{-2t} \\ -4e^{-2t} + 4te^{-2t} \\ 12e^{-2t} - 8te^{-2t} \end{pmatrix} + k_3 \mathbf{x}^{(2)}(t) + k_4 \mathbf{x}^{(3)}(t).$$

Now this must be the general solution that eluded us in part 5 (because we did not have a fundamental set of solutions). So, it would appear (and correctly so) that a fourth L.I. solution is

$$\mathbf{x}^{(4)}(t) = \begin{pmatrix} te^{-2t} \\ e^{-2t} - 2te^{-2t} \\ -4e^{-2t} + 4te^{-2t} \\ 12e^{-2t} - 8te^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} t \\ 1 - 2t \\ -4 + 4t \\ 12 - 8t \end{pmatrix} = e^{-2t} \left[\begin{pmatrix} 0 \\ 1 \\ -4 \\ 12 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 4 \\ -8 \end{pmatrix} \right].$$

We expected this fourth solution to be associated (in some fashion) with the eigenvalue $\lambda = -2$. Based on our experience with finding another L.I. solution for scalar equations in situations where we have a repeated root of the characteristic equation, we might have guessed that the fourth L.I. solution was

$$te^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \\ -8 \end{pmatrix}$$

(i.e., t times the already-found solution $\mathbf{x}^{(1)}$ associated with the eigenvalue $\lambda = -2$). In fact this guess is not even a solution of the matrix DE. (You should verify this!) Nevertheless, this naive guess does appear as part of $\mathbf{x}^{(4)}$, along with an additional vector $(0, 1, -4, 12)$. The fact that the individual components of each solution $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ and $\mathbf{x}^{(4)}$ correspond to a solution of the original 4th-order DE and its first three derivatives ($x^{(4)}$ corresponds to $y = te^{-2t}$) explains the presence of this additional vector for our particular problem. (Do you see why?) What is yet to be explained is how, had we not solved the matrix DE already using Chapter 3 methods, we would have found this additional vector. We will need to know how, since matrices can indeed have repeated eigenvalues with a correspondingly deficient set of eigenvectors (i.e., not as many L.I. ones as the multiplicity of the eigenvalue, as was the case here), and, as we said before, many (in fact most) matrix DEs cannot be converted to an nth-order (scalar) DE.