MATH 231, Worksheet
Date: May 4, 2018

Find the inverse Laplace transform for each function.

1. $F(s) = \dfrac{e^{-\pi s/2}}{s^2+9}$

   Answer: First, we deal with $(s^2+9)^{-1}$; that is, $F(s)$ without its exponential factor. We have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{1}{3}\sin(3t).$$

   The exponential factor in $F(s)$ calls for use of the **1st shifting theorem**, yielding

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{3}u\left(t-\frac{\pi}{2}\right)\sin\left(3\left(t-\frac{\pi}{2}\right)\right).$$

2. $F(s) = \dfrac{1}{s^2(s^2+4)}$

   Answer: The partial fraction expansion for $F(s)$ is

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+4}.$$

   Multiplying through by the common denominator yields the equation

$$\begin{aligned} 1 &= As(s^2+4)+B(s^2+4)+(Cs+D)s^2 \\ &= (A+C)s^3+(B+D)s^2+4As+4B. \end{aligned}$$

   Since this must hold for all real s, the coefficients of the various powers of s must be equal:

$$\begin{array}{rrcl} s^3: & 0 &=& A+C \\ s^2: & 0 &=& B+D \\ s^1: & 0 &=& 4A \\ s^0: & 1 &=& 4B \end{array} \quad\Rightarrow\quad \begin{array}{rcl} C &=& 0 \\ D &=& -1/4 \\ A &=& 0 \\ B &=& 1/4 \end{array}$$

   Thus,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{4}t-\frac{1}{8}\sin(2t).$$

3. $F(s) = \dfrac{s}{s^2+6s+11}$

   Answer: First, note that $s^2+6s+11$ is an irreducible quadratic, having nonreal roots. Thus, we complete the square:

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+11}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{(s^2+6s+9)+2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+3)-3}{(s+3)^2+2}\right\} \\ &= \mathcal{L}^{-1}\left\{\left.\frac{s}{s^2+2}\right|_{s\mapsto s+3} - \left.\frac{3}{s^2+2}\right|_{s\mapsto s+3}\right\} \\ &= \mathcal{L}^{-1}\left\{\left.\frac{s}{s^2+2}\right|_{s\mapsto s+3}\right\} - \frac{3}{\sqrt{2}}\mathcal{L}^{-1}\left\{\left.\frac{\sqrt{2}}{s^2+2}\right|_{s\mapsto s+3}\right\} \\ &= e^{-3t}\cos(\sqrt{2}t) - \frac{3}{\sqrt{2}}e^{-3t}\sin(\sqrt{2}t). \end{aligned}$$

4. $F(s) = e^{-s}\dfrac{s}{s^2+6s+11}$

Answer: Building on our answer to the previous exercise, we have

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{s}{s^2+6s+11}\right\} \;=\; u(t-1)\left[e^{-3(t-1)}\cos(\sqrt{2}(t-1)) - \frac{3}{\sqrt{2}}e^{-3(t-1)}\sin(\sqrt{2}(t-1))\right].$$

5. $F(s) = \dfrac{1}{(s-1)^3} + \dfrac{1}{s^2+2s-8}$

Answer: We first perform partial fraction decomposition on the second term. [Note that such a decomposition of the first term *cannot* make it simpler.]

$$\frac{1}{s^2+2s-8} \;=\; \frac{1}{(s-2)(s+4)} \;=\; \frac{A}{s-2} + \frac{B}{s+4}.$$

We find that $A = 1/6$ and $B = -1/6$. Thus,

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3} + \frac{1}{6}\cdot\frac{1}{s-2} - \frac{1}{6}\cdot\frac{1}{s+4}\right\} \\
&= \frac{1}{2}\mathcal{L}^{-1}\left\{\left.\frac{2}{s^3}\right|_{s\mapsto s-1}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} \\
&= \frac{1}{2}\,t^2e^t + \frac{1}{6}e^{2t} - \frac{1}{6}e^{-4t},
\end{aligned}$$

where, in the final step, we have used the **2$^{\text{nd}}$ shifting theorem** on the first of the three Laplace transforms.

6. $F(s) = e^{-2s}\dfrac{1}{(s-1)^3} + e^{-s}\dfrac{1}{s^2+2s-8}$

Answer: First, if we look carefully back at the work of the previous problem, we note that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} \;=\; \frac{1}{2}\,t^2e^t \qquad \text{and} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-8}\right\} \;=\; \frac{1}{6}(e^{2t}-e^{-4t}).$$

By the 1$^{\text{st}}$ shifting theorem, we obtain

$$\mathcal{L}^{-1}\{F(s)\} \;=\; \frac{1}{2}u(t-2)(t-2)^2e^{t-2} + \frac{1}{6}u(t-1)\left(e^{2(t-1)} - e^{-4(t-1)}\right).$$