

MATH 231A  
Solutions to PS 10

Solution to Problem \*\*

In one sense, this problem may be written in matrix vector form, as in

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2-x \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

But if we are going to demand that the coefficient matrix  $A$  be a function of  $t$  only (as in the 1st-order matrix-vector DE  $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{b}(t)$ ), then the matrix

$$\begin{pmatrix} 3 & 0 \\ 0 & 2-x \end{pmatrix}$$

does not fit the bill, and no other coefficient matrix for the system does. It is this latter sense in which the first question was asked, and hence the answer is “no”.

Turning to the solution of the system, we note first that the DE  $x' = 3x$  is both linear and separable. We may even be able to guess the general solution to this equation, which is

$$x(t) = c_1 e^{3t}.$$

While the 2nd DE looks to be nonlinear (because of the product between dependent variables  $x$  and  $y$ ), it becomes linear when we substitute in our expression for  $x(t)$ :

$$\begin{aligned} y' = (2-x)y &\Rightarrow y' = (2 - c_1 e^{3t})y \\ &\Rightarrow \frac{dy}{y} = (2 - c_1 e^{3t}) dt \\ &\Rightarrow \ln |y| = \int (2 - c_1 e^{3t}) dt = 2t - \tilde{c}_1 e^{3t} + c_2 \\ &\Rightarrow y(t) = \tilde{c}_2 e^{2t - \tilde{c}_1 e^{3t}}. \end{aligned}$$

Solution to Problem \*\*\*\*(b)

$$\begin{aligned} \det A - \lambda I &= \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda \\ -1 & 1 \end{vmatrix} \\ &= -(\lambda^3 - 3\lambda^2 + 6\lambda - 4) \\ &= -(\lambda - 1)(\lambda^2 - 2\lambda + 4). \end{aligned}$$

Thus, the eigenvalues of  $A$  are

$$\lambda = 1 \quad \text{and} \quad \lambda = \frac{1}{2}(2 \pm \sqrt{-12}) = 1 \pm i\sqrt{3}.$$

To find the associated eigenvectors, we proceed as usual. For  $\lambda = 1$ , we solve  $(A - I)\mathbf{v} = \mathbf{0}$ :

$$\begin{array}{l} \left( \begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 \\ \sim \end{array} \quad \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array} \right) \\ \\ \quad \quad \quad \begin{array}{l} \mathbf{r}_1 + \mathbf{r}_3 \rightarrow \mathbf{r}_3 \\ \sim \end{array} \quad \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right) \\ \\ \quad \quad \quad \begin{array}{l} -\mathbf{r}_2 \rightarrow \mathbf{r}_2 \\ \sim \mathbf{r}_2 + \mathbf{r}_3 \rightarrow \mathbf{r}_3 \end{array} \quad \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right). \end{array}$$

The third column has no pivot, so we take  $v_3$  to be free:

$$v_3 = \alpha, \quad \text{which implies} \quad v_2 = \alpha \quad \text{and} \quad v_1 = \alpha.$$

Thus, the eigenvectors associated with eigenvalue  $\lambda = 1$  are scalar multiples of the vector  $(1, 1, 1)$ .

Before turning to the nonreal (complex) eigenvalues, we note something about doing arithmetic with complex numbers. In particular, we note that division by the complex number  $(a + ib)$  is equivalent to multiplication by  $a/(a^2 + b^2) - ib/(a^2 + b^2)$ ; that is,

$$\frac{1}{a + ib} = \frac{1}{a + ib} \cdot \frac{a - ib}{a - ib} = \frac{a - ib}{a^2 + b^2} = \left( \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \right).$$

If, more particularly, we are dividing by the imaginary number  $ib$ , then it is equivalent to multiply by  $-i/b$ .

Turning to the eigenvalue  $\lambda = 1 + i\sqrt{3}$ , we solve  $[A - (1 + i\sqrt{3})I]\mathbf{v} = \mathbf{0}$ :

$$\begin{array}{l} \left( \begin{array}{ccc|c} -i\sqrt{3} & -1 & 1 & 0 \\ 1 & -i\sqrt{3} & -1 & 0 \\ -1 & 1 & -i\sqrt{3} & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 \\ \sim \end{array} \quad \left( \begin{array}{ccc|c} 1 & -i\sqrt{3} & -1 & 0 \\ -i\sqrt{3} & -1 & 1 & 0 \\ -1 & 1 & -i\sqrt{3} & 0 \end{array} \right) \\ \\ \quad \quad \quad \begin{array}{l} i\sqrt{3}\mathbf{r}_1 + \mathbf{r}_2 \rightarrow \mathbf{r}_2 \\ \sim \end{array} \quad \left( \begin{array}{ccc|c} 1 & -i\sqrt{3} & -1 & 0 \\ 0 & 2 & 1 - i\sqrt{3} & 0 \\ 0 & 1 - i\sqrt{3} & -1 - i\sqrt{3} & 0 \end{array} \right) \\ \\ \quad \quad \quad \mathbf{r}_1 + \mathbf{r}_3 \rightarrow \mathbf{r}_3 \quad \left( \begin{array}{ccc|c} 1 & -i\sqrt{3} & -1 & 0 \\ 0 & 2 & 1 - i\sqrt{3} & 0 \\ 0 & 1 - i\sqrt{3} & -1 - i\sqrt{3} & 0 \end{array} \right) \\ \\ \quad \quad \quad \begin{array}{l} (1/2)\mathbf{r}_2 \rightarrow \mathbf{r}_2 \\ \sim \end{array} \quad \left( \begin{array}{ccc|c} 1 & -i\sqrt{3} & -1 & 0 \\ 0 & 1 & 1/2 - i\sqrt{3}/2 & 0 \\ 0 & 1 - i\sqrt{3} & -1 - i\sqrt{3} & 0 \end{array} \right) \\ \\ \quad \quad \quad \begin{array}{l} i\sqrt{3}\mathbf{r}_2 + \mathbf{r}_1 \rightarrow \mathbf{r}_1 \\ \sim \end{array} \quad \left( \begin{array}{ccc|c} 1 & 0 & 1/2 + i\sqrt{3}/2 & 0 \\ 0 & 1 & 1/2 - i\sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \\ \\ \quad \quad \quad \mathbf{r}_3 - (1 - i\sqrt{3})\mathbf{r}_2 \rightarrow \mathbf{r}_3 \quad \left( \begin{array}{ccc|c} 1 & 0 & 1/2 + i\sqrt{3}/2 & 0 \\ 0 & 1 & 1/2 - i\sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

Again taking  $v_3 = \alpha$ , we have

$$v_1 = \left( -\frac{1}{2} - i\frac{\sqrt{3}}{2} \right) \alpha, \quad \text{and} \quad v_2 = \left( -\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) \alpha;$$

that is, eigenvectors associated with the eigenvalue  $\lambda = 1 + i\sqrt{3}$  are all scalar multiples of the vector  $(-\frac{1}{2} - i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, 1)$  or, equivalently, all scalar multiples of  $(1 + i\sqrt{3}, 1 - i\sqrt{3}, -2)$ .

We can repeat the above procedure in order to find the eigenvectors associated with the eigenvalue  $\lambda = 1 - i\sqrt{3}$ , but as explained in the linear algebra handout, matrices with real entries (as opposed to complex ones) will not only have eigenvalues that come in complex-conjugate pairs (as were the two eigenvalues  $\lambda = 1 + i\sqrt{3}$  and  $\lambda = 1 - i\sqrt{3}$ ), but the eigenvectors associated with these eigenvalues also are complex conjugates in a certain sense. That is, writing the eigenvectors associated with  $1 + i\sqrt{3}$  as scalar multiples of

$$(1 + i\sqrt{3}, 1 - i\sqrt{3}, -2) = (1, 1, -2) + i(\sqrt{3}, -\sqrt{3}, 0),$$

the eigenvectors associated with  $1 - i\sqrt{3}$  will be all scalar multiples of

$$(1, 1, -2) - i(\sqrt{3}, -\sqrt{3}, 0) = (1 - i\sqrt{3}, 1 + i\sqrt{3}, -2).$$