

MATH 231A
Solutions to PS 8

** (a)

$$\begin{aligned}
 \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 3 & -1 \end{array} \right) & \xrightarrow{-2\mathbf{r}_1 + \mathbf{r}_2 \rightarrow \mathbf{r}_2} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 5 & -3 \end{array} \right) \\
 & \sim \\
 & \xrightarrow{(1/5)\mathbf{r}_2 \rightarrow \mathbf{r}_2} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -3/5 \end{array} \right) \\
 & \sim \\
 & \xrightarrow{\mathbf{r}_1 + \mathbf{r}_2 \rightarrow \mathbf{r}_1} \left(\begin{array}{cc|c} 1 & 0 & 2/5 \\ 0 & 1 & -3/5 \end{array} \right).
 \end{aligned}$$

Thus, the solution is $(x, y) = (2/5, -3/5)$.

(b)

$$\begin{aligned}
 \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 1 & 1 & -1 & -3 \\ -1 & 1 & 1 & 1 \end{array} \right) & \xrightarrow{\mathbf{r}_2 - \mathbf{r}_1 \rightarrow \mathbf{r}_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 2 & -2 & -8 \\ 0 & 0 & 2 & 6 \end{array} \right) \\
 & \sim \\
 & \xrightarrow{\mathbf{r}_3 + \mathbf{r}_1 \rightarrow \mathbf{r}_3} \\
 & \xrightarrow{(1/2)\mathbf{r}_2 \rightarrow \mathbf{r}_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{array} \right) \\
 & \sim \\
 & \xrightarrow{(1/2)\mathbf{r}_3 \rightarrow \mathbf{r}_3} \\
 & \xrightarrow{\mathbf{r}_1 + \mathbf{r}_2 \rightarrow \mathbf{r}_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right) \\
 & \sim \\
 & \xrightarrow{\mathbf{r}_2 + \mathbf{r}_3 \rightarrow \mathbf{r}_2}
 \end{aligned}$$

Thus, $(x, y, z) = (1, -1, 3)$.

(c)

$$\begin{aligned}
 \left(\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \\ 1 & 0 & -1 & 0 \end{array} \right) & \xrightarrow{\mathbf{r}_1 \leftrightarrow \mathbf{r}_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -1 & 1 & 2 & 2 \\ 3 & 1 & 1 & 1 \end{array} \right) \\
 & \sim \\
 & \xrightarrow{\mathbf{r}_1 + \mathbf{r}_2 \rightarrow \mathbf{r}_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 4 & 1 \end{array} \right) \\
 & \sim \\
 & \xrightarrow{-3\mathbf{r}_1 + \mathbf{r}_3 \rightarrow \mathbf{r}_3} \\
 & \xrightarrow{\mathbf{r}_3 - \mathbf{r}_2 \rightarrow \mathbf{r}_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & -1 \end{array} \right) \\
 & \sim \\
 & \xrightarrow{(1/3)\mathbf{r}_3 \rightarrow \mathbf{r}_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1/3 \end{array} \right) \\
 & \sim
 \end{aligned}$$

$$\begin{array}{l} \mathbf{r}_1 + \mathbf{r}_3 \rightarrow \mathbf{r}_1 \\ \sim \\ \mathbf{r}_2 - \mathbf{r}_3 \rightarrow \mathbf{r}_2 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & -1/3 \end{array} \right)$$

Thus, $(x, y, z) = (-1/3, 7/3, -1/3)$.

*** (a) Expanding initially along the first row, we have

$$\begin{aligned} \left| \begin{array}{cccc} 0 & 2 & 1 & 3 \\ -1 & -1 & 2 & 0 \\ 2 & -1 & 0 & 1 \\ -1 & 7 & -5 & 1 \end{array} \right| &= -2 \left| \begin{array}{ccc} -1 & 2 & 0 \\ 2 & 0 & 1 \\ -1 & -5 & 1 \end{array} \right| + \left| \begin{array}{ccc} -1 & -1 & 0 \\ 2 & -1 & 1 \\ -1 & 7 & 1 \end{array} \right| - 3 \left| \begin{array}{ccc} -1 & -1 & 2 \\ 2 & -1 & 0 \\ -1 & 7 & -5 \end{array} \right| \\ &= -2 \left(- \left| \begin{array}{cc} 0 & 1 \\ -5 & 1 \end{array} \right| - 2 \left| \begin{array}{cc} 2 & 1 \\ -1 & 1 \end{array} \right| \right) + \left(- \left| \begin{array}{cc} -1 & 1 \\ 7 & 1 \end{array} \right| + \left| \begin{array}{cc} 2 & 1 \\ -1 & 1 \end{array} \right| \right) \\ &\quad - 3 \left(-2 \left| \begin{array}{cc} -1 & 2 \\ 7 & -5 \end{array} \right| - \left| \begin{array}{cc} -1 & 2 \\ -1 & -5 \end{array} \right| \right) \\ &= -2[-5 - 2(3)] + 8 + 3 - 3[-2(-9) - 7] \\ &= 0. \end{aligned}$$

(b) Since the coefficient matrix has zero determinant, the system either has no solution or infinitely many. However, since it is clear that $(x, y, z, w) = (0, 0, 0, 0)$ is a solution, the system must have infinitely many solutions.

(c) Since the coefficient matrix has zero determinant, the system either has no solution or infinitely many. This time, there is no obvious solution (in contrast to part (b)), so we'll draw no other conclusion than this at this time.

(d)

$$\begin{array}{l} \left(\begin{array}{cccc|c} 0 & 2 & 1 & 3 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 1 & 0 \\ -1 & 7 & -5 & 1 & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{r}_1 \leftrightarrow (-\mathbf{r}_2) \\ \sim \\ 2\mathbf{r}_2 + \mathbf{r}_3 \rightarrow \mathbf{r}_3 \\ \mathbf{r}_4 - \mathbf{r}_2 \rightarrow \mathbf{r}_4 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & -3 & 4 & 1 & 0 \\ 0 & 8 & -7 & 1 & 0 \end{array} \right) \\ \\ \begin{array}{l} (1/2)\mathbf{r}_2 \leftrightarrow \mathbf{r}_2 \\ \sim \\ 3\mathbf{r}_2 + 2\mathbf{r}_3 \rightarrow \mathbf{r}_3 \\ \mathbf{r}_4 - 4\mathbf{r}_2 \rightarrow \mathbf{r}_4 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 1/2 & 3/2 & 0 \\ 0 & 0 & 11 & 11 & 0 \\ 0 & 0 & -11 & -11 & 0 \end{array} \right) \\ \\ \begin{array}{l} (1/11)\mathbf{r}_3 \leftrightarrow \mathbf{r}_3 \\ \sim \\ \mathbf{r}_3 + \mathbf{r}_4 \rightarrow \mathbf{r}_4 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 1/2 & 3/2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \\ \\ \sim \quad \left(\begin{array}{cccc|c} 1 & 0 & -5/2 & -3/2 & 0 \\ 0 & 1 & 1/2 & 3/2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \\ \\ \mathbf{r}_1 - \mathbf{r}_2 \rightarrow \mathbf{r}_1 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 + (5/2)\mathbf{r}_3 \rightarrow \mathbf{r}_1 \\ \sim \\ \mathbf{r}_2 - (1/2)\mathbf{r}_3 \rightarrow \mathbf{r}_2 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

There are pivots in the first three columns, but not in the 4th column. So, we take w to be a free variable, assigning it the value α (an arbitrary constant). Then

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\alpha \\ -\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

That is, all solutions are arbitrary multiples of the vector $(-1, -1, -1, 1)$ (or, equivalently, arbitrary multiples of $(1, 1, 1, -1)$).

(e)

$$\begin{array}{l} \left(\begin{array}{cccc|c} 0 & 2 & 1 & 3 & 4 \\ -1 & -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 1 & 5 \\ -1 & 7 & -5 & 1 & -6 \end{array} \right) \\ \mathbf{r}_1 \leftrightarrow (-\mathbf{r}_2) \\ \sim \\ 2\mathbf{r}_2 + \mathbf{r}_3 \rightarrow \mathbf{r}_3 \\ \mathbf{r}_4 - \mathbf{r}_2 \rightarrow \mathbf{r}_4 \\ \\ (1/2)\mathbf{r}_2 \rightarrow \mathbf{r}_2 \\ \sim \\ 3\mathbf{r}_2 + 2\mathbf{r}_3 \rightarrow \mathbf{r}_3 \\ \mathbf{r}_4 - 4\mathbf{r}_2 \rightarrow \mathbf{r}_4 \\ \\ (1/11)\mathbf{r}_3 \rightarrow \mathbf{r}_3 \\ \sim \\ \mathbf{r}_3 + \mathbf{r}_4 \rightarrow \mathbf{r}_4 \\ \\ \sim \\ \mathbf{r}_1 - \mathbf{r}_2 \rightarrow \mathbf{r}_1 \\ \\ \mathbf{r}_1 + (5/2)\mathbf{r}_3 \rightarrow \mathbf{r}_1 \\ \sim \\ \mathbf{r}_2 - (1/2)\mathbf{r}_3 \rightarrow \mathbf{r}_2 \end{array} \left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 0 \\ 0 & 2 & 1 & 3 & 4 \\ 0 & -3 & 4 & 1 & 5 \\ 0 & 8 & -7 & 1 & -6 \end{array} \right) \\ \\ \left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 1/2 & 3/2 & 2 \\ 0 & 0 & 11 & 11 & 22 \\ 0 & 0 & -11 & -11 & -22 \end{array} \right) \\ \\ \left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 1/2 & 3/2 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \\ \\ \left(\begin{array}{cccc|c} 1 & 0 & -5/2 & -3/2 & -2 \\ 0 & 1 & 1/2 & 3/2 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \\ \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

As before, w is a free variable, and we assign it the value α . Then

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 - \alpha \\ 1 - \alpha \\ 2 - \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

That is, all solutions are the sum of the vector $(3, 1, 2, 0)$ and an arbitrary multiple of $(-1, -1, -1, 1)$. Note that, having solved the system, we may now amend our answer to part (c) and say that there are infinitely many solutions.

(f) In Exercise 26, Section 2.4, we considered the linear ODE

$$y' + p(t)y = g(t)$$

and found solutions to be of the form

$$y_1(t) + cy_2(t)$$

(the sum of a particular function y_1 and an arbitrary multiple of y_2), where $y_2(t)$ was one of the infinitely-many solutions to the special case where $g(t) \equiv 0$. The situation here is similar. Solutions in part (e) are made up of a sum of a particular vector and arbitrary multiples of a solution to the system in part (d), where $\mathbf{b} = \mathbf{0}$ (the zero vector).

Solutions to PS 9

* (a) First, we compute the determinant

$$\begin{aligned} \det \left(\left(\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \right) &= \begin{vmatrix} 3 - \lambda & 2 \\ 1 & 4 - \lambda \end{vmatrix} \\ &= (3 - \lambda)(4 - \lambda) - 2 \\ &= \lambda^2 - 7\lambda + 10 \\ &= (\lambda - 5)(\lambda - 2). \end{aligned}$$

We see that this determinant is zero precisely when $\lambda = 2, 5$. These are the eigenvalues. For $\lambda = 2$, we find the associated eigenvectors solving the system of equations

$$\left(\left(\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Using Gaussian elimination, we have

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array} \right) \xrightarrow{\mathbf{r}_2 - \mathbf{r}_1 \rightarrow \mathbf{r}_2} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right),$$

showing that y is a free variable. Setting $y = \alpha$, we have that the eigenvectors associated with $\lambda = 2$ are of the form

$$(x, y) = \alpha(-2, 1)$$

(that is, all scalar multiples of $(-2, 1)$). For $\lambda = 5$, we use Gaussian elimination on the augmented matrix

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array} \right) \xrightarrow{2\mathbf{r}_2 + \mathbf{r}_1 \rightarrow \mathbf{r}_2} \left(\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

Again, y is a free variable, and setting $y = \alpha$, we have that the eigenvectors associated with $\lambda = 5$ are of the form

$$(x, y) = \alpha(1, 1)$$

(that is, all scalar multiples of $(1, 1)$).

(b) Proceeding as above, we compute the determinant

$$\begin{aligned} \det \left(\left(\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \right) &= \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} \\ &= (3-\lambda)(1-\lambda) + 1 \\ &= \lambda^2 - 4\lambda + 4 \\ &= (\lambda - 2)^2. \end{aligned}$$

We see that $\lambda = 2$ is a double root of this determinant — that is, it is a repeated eigenvalue. To find the associated eigenvectors, we work with the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array} \right) \begin{array}{l} \mathbf{r}_2 + \mathbf{r}_1 \rightarrow \mathbf{r}_2 \\ \sim \\ \mathbf{r}_2 + \mathbf{r}_1 \rightarrow \mathbf{r}_2 \end{array} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

Taking y to be free (so $y = \alpha$), we have that all associated eigenvectors have the form

$$(x, y) = \alpha(1, -1).$$

(c) Once again, we compute the determinant

$$\begin{aligned} \det \left(\left(\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \right) &= \begin{vmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda) + 2 \\ &= \lambda^2 - 4\lambda + 5. \end{aligned}$$

This quadratic polynomial does not factor easily, and so we use the quadratic formula to find its zeros:

$$\lambda = \frac{1}{2}(4 \pm \sqrt{-4}) = \frac{1}{2}(4 \pm 2i) = 2 \pm i.$$

This is our first instance of nonreal eigenvalues, but this is by no means rare. To find associated eigenvectors, we proceed as before. For $\lambda = 2+i$, we work with the augmented matrix

$$\left(\begin{array}{cc|c} -1-i & -1 & 0 \\ 2 & 1-i & 0 \end{array} \right) \begin{array}{l} (1-i)\mathbf{r}_1 + \mathbf{r}_2 \rightarrow \mathbf{r}_2 \\ \sim \\ (-1+i)\mathbf{r}_1 \rightarrow \mathbf{r}_1 \end{array} \left(\begin{array}{cc|c} 2 & 1-i & 0 \\ 0 & 0 & 0 \end{array} \right).$$

Taking $y = \alpha$, we have that the eigenvectors associated with $\lambda = 2+i$ are of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} (-1+i)/2 \\ 1 \end{pmatrix} = \tilde{\alpha} \begin{pmatrix} -1+i \\ 2 \end{pmatrix} = \tilde{\alpha} \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].$$

(that is, all scalar multiples of $(-1, 2) + i(1, 0)$). As it turns out, since the entries of our coefficient matrix were all real numbers, we do not have to carry out similar work as above for the eigenvalue $\lambda = 2-i$. Its eigenvectors will be of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \tilde{\alpha} \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} - i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].$$