Peer Questions for Section 8.7

Read this material prior to class on Mon., Oct. 21, attempting to answer the questions below. In your group (minimum of two people per), discuss your responses to the following questions. Rotate (again) the role of "group scribe", a person who should submit your group’s responses, using the web form below, by 5 pm, Mon., Oct. 21.

1. Consider the polynomial \( f(x) = 5 - 3x + 2x^2 + x^3 \) which, like all polynomials, is an example of a power series. As written, it is currently centered at 0, but it can be rewritten so as to be centered at \( x = 2 \) (or any other number) using simple algebra:

\[
5 - 3x + 2x^2 + x^3 = 5 - 3x + 2x^2 + (x - 2)^3 + 6x^2 - 12x + 8
= 13 - 15x + 8x^2 + (x - 2)^3
= 13 - 15x + 8(x - 2)^2 + (x - 2)^3 + 32x - 32
= -19 + 17x + 8(x - 2)^2 + (x - 2)^3
= -19 + 17(x - 2) + 8(x - 2)^2 + (x - 2)^3 + 34
= 15 + 17(x - 2) + 8(x - 2)^2 + (x - 2)^3.
\]

That is, when written as a power series \( \sum_n c_n(x - 2)^n \), centered at 2, the coefficients are

\[
c_0 = 15, \quad c_1 = 17, \quad c_2 = 8, \quad c_3 = 1, \quad c_4 = c_5 = \cdots = c_n = \cdots = 0.
\]

Find these coefficients via the formula

\[
c_n = \frac{f^{(n)}(2)}{n!},
\]

with \( a = 2 \).

2. Find the formulas for \( T_0(x), T_1(x), T_2(x), T_3(x) \) and \( T_4(x) \), the Taylor polynomials of \( \cos x \) (centered) at zero. Guess the formula for \( T_{2n}(x) \). How does \( T_{2n+1}(x) \) differ from \( T_{2n}(x) \)?

3. True or False.

(a) The 1st-degree Taylor polynomial of \( f(x) = 5 - 3x + 2x^2 + x^3 \) at 2 is \( T_1(x) = 17x - 19 \).

(b) The 1st-degree Taylor polynomial of the same function \( f \) as in part (a) at 0 is \( T_1(x) = -3x + 5 \).

(c) The 1st-degree Taylor polynomial of \( f \) at \( a \) is really the tangent line to the graph of \( y = f(x) \) at \( (a, f(a)) \).

(d) If \( p(x) \) is a polynomial of degree \( k \), and \( T_0(x), T_1(x), T_2(x), T_3(x), \ldots \) represent the 0th, 1st, 2nd, 3rd, etc.-degree Taylor polynomials of \( p \) at \( a \), then \( T_k(x) = T_{k+1}(x) = T_{k+2}(x) = \ldots \).

(e) Every function \( f \) that includes \( x = a \) in its domain has an \( n \)-th-degree Taylor polynomial \( T_n(x) \) at \( a \), regardless of the choices of \( n \) and \( a \).
4. When your function $f$ has derivatives of all orders, you can use formula (1) to find values for $c_n$ of a power series centered at $a$: \[ \sum_{n=0}^{\infty} c_n (x - a)^n. \] There is no assurance, however, that the power series which results, and the $f$ which was used to generate that series, will be equal at any choice of $x \neq a$, even if the radius of convergence of the power series $R > 0$. Our desire, of course, is that they be equal, and for many functions they are. The first example in the section for which this is proved is $f(x) = e^x$, which equals its (Maclaurin) series expansion \[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \] (2) Look carefully at the logical structure of the argument in the text. At what stage was it finally proved that the equation (2) holds for all real numbers $x$? (Choose one.)

(a) When it was established that the radius of convergence of the series is $R = \infty$.

(b) When it was demonstrated that the limit \[ \lim_{n \to \infty} \left( e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right) = 0, \quad \text{for all real } x. \]

(c) When it was shown that $f^{(n)}(0) = 1$ for each $n = 0, 1, 2, \ldots$, thereby establishing that each series coefficient $c_n = \frac{1}{n!}$.

5. Identify one item (a concept, a step in an example, a statement, etc.) from this reading assignment you found difficult or confusing.