Definition: A function (or function of $n$ variables) $f$ is a rule that assigns to each ordered $n$-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ in a certain set $D$ a real number $f(x_1, x_2, \ldots, x_n)$. The set $D$ is called the domain of the function.

Example: Most real-life functions are, in fact, functions of multiple variables. Here are some:

1. $v(r, h) = \pi r^2 h$
2. $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
3. $g(m_1, m_2, R) = G m_1 m_2 / R^2$ ($G$ is a constant)
4. $P(n, T, V) = nRT/V$ ($R$ is a constant)

Of course, one can hold fixed the values of all but one of the input variables and thereby create a function of a single variable. For instance, the way the volume of a right-circular cylinder whose height is 3 varies with its radius is given by the formula

$$V(r) = 3\pi r^2, \quad r \geq 0.$$ 

Graphing

Functions of a single variable

To graph a function of a single variable requires two coordinate axes. When we write $y = f(x)$, it is implied that $x$ is a possible input and the $y$-value is the corresponding output. We think of the domain (the set of all possible inputs) of $f$ as consisting of some part of the real line, the graph of $f$ (often called a “curve”) as having a point at location $(x, f(x))$ for each $x$ in the domain of $f$. Keep in mind that, given an arbitrary equation involving $x$ and $y$, it is not always the case that

(i) we want to make $y$ be the dependent variable, and

(ii) if we do solve for $y$, the result is a function.

Example: $x^2 + y^2 = 4$
Functions of multiple variables

We have the following analogies for functions of multiple variables:

• When nothing explicit is said about the inputs to a function of multiple variables, we take the domain to be as inclusive as possible.

  Examples:
  
  \[ f(x, y) = \sqrt{xy} \]
  \[ f(x, y) = xy(x^2 + y)^{-1} \]

• The graphs of functions of \( n \) variables are \( n \)-dimensional objects drawn in a coordinate frame involving \( (n + 1) \) mutually-perpendicular coordinate axes. (Think of a curve which is the graph of \( y = f(x) \) as a 1-dimensional object weaving through 2-dimensional space.)

As a corollary: It is not possible to produce the graph of a function of 3 or more variables. A possible work-around: level sets.

**Definition:** Let \( f \) be a function of \( n \) variables, and \( c \) be a real number. The set of all \( n \)-tuples \( (x_1, \ldots, x_n) \) for which \( f(x_1, \ldots, x_n) = c \) is called the **c level set of \( f \)**.

Examples:

\[ f(x, y) = y^2 - x^2 \]
\[ f(x, y, z) = z - x^2 - 2y^2 \]

• Not every equation involving \( x, y \) and \( z \) yields \( z \) as a single function of \( x \) and \( y \).

  Examples:
  
  \[ x - y^2 - z^2 = 0 \]
  \[ x^2 + y^2 + z^2 = 4 \]

• One may assume a missing variable is implied and takes on all real values.

  **Example:** The meaning of \( x = 1 \) in 1, 2 and 3 dimensions.

  One may need more than one equation/inequality to describe certain regions of space.

  **Example:** \( x^2 + (y - 1)^2 \leq 1, \quad z = -1 \)