While power series are allowed to have nonzero numbers as centers, for today’s results we will assume all power series we discuss are centered about \( x = 0 \); that is, are of the form
\[
\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots.
\]
(1)
We will also assume each has a positive radius of convergence \( R > 0 \), so that the series converges at least for those \( x \) satisfying \(-R < x < R\).

Differentiation of Power Series about \( x = 0 \)

Theorem (Term-by-Term Differentiation, p. 549): Let \( f(x) \) take the form of the power series in (1), with radius of convergence \( R > 0 \). Then the series
\[
\sum_{n=1}^{\infty} n c_n x^{n-1}
\]
converges for all \( x \) satisfying \(-R < x < R\), and
\[
f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad \text{for all } x \text{ satisfying } -R < x < R.
\]

Remarks:
- Since the hypotheses of this theorem now apply to \( f'(x) \), we can continue to differentiate the series to find derivatives of \( f \) of all orders, convergent at least on the interval \(-R < x < R\). For instance,

\[
f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = (2 \cdot 1)c_2 + (3 \cdot 2)c_3 x + (4 \cdot 3)c_4 x^2 + \cdots
\]
\[
f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) c_n x^{n-3} = (3 \cdot 2 \cdot 1)c_3 + (4 \cdot 3 \cdot 2)c_4 x + (5 \cdot 4 \cdot 3)c_5 x^2 + \cdots
\]
\[
\vdots
\]
\[
f^{(j)}(x) = \sum_{n=j}^{\infty} n(n-1)(n-2) \cdots (n-j+1) c_n x^{n-j}
\]
\[
= j! c_j + [(j+1)j \cdots 2] c_{j+1} x + [(j+2)(j+1) \cdots 3] c_{j+2} x^2 + \cdots.
\]
• The easiest place to evaluate a power series is at its center. In particular, if \( f \) has the form \((1)\), we may evaluate \( f \) and all of its derivatives at zero to get:

\[
\begin{align*}
  f(0) &= c_0, \\
  f'(0) &= c_1, \\
  f''(0) &= (2 \cdot 1) c_2 \quad \Rightarrow \quad c_2 = \frac{1}{2} f''(0), \\
  f'''(0) &= (3 \cdot 2 \cdot 1) c_3 \quad \Rightarrow \quad c_3 = \frac{1}{3!} f'''(0),
\end{align*}
\]

and so on. So, we arrive at the following relationship between the coefficients \( c_j \) and the derivatives of \( f \) at the center:

**Corollary:** Let \( f(x) \) be defined by the power series \((1)\). Then

\[
c_n = \frac{f^{(n)}(0)}{n!}, \quad \text{for all } n = 0, 1, 2, \ldots.
\]

**Example:** We know that \( f(x) = (1 - x)^{-1} \) has the power series representation \( \sum_{n=0}^{\infty} x^n \) for \( x \) in the interval \((-1, 1)\). By the term-by-term differentiation theorem, \( f'(x) = (1 - x)^{-2} \) has the power series representation

\[
\frac{1}{(1 - x)^2} = \frac{d}{dx} \left( \sum_{n=0}^{\infty} x^n \right) = \frac{d}{dx} \left( 1 + x + x^2 + x^3 + \cdots \right)
\]

\[
= \frac{d}{dx} (1) + \frac{d}{dx} (x) + \frac{d}{dx} (x^2) + \frac{d}{dx} (x^3) + \cdots \quad \text{(the theorem justifies this step)}
\]

\[
= 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots
\]

\[
= \sum_{n=1}^{\infty} nx^{n-1},
\]

with this series representation holding at least for \(-1 < x < 1\).

**Integration of Power Series about \( x = 0 \)**

If we can differentiate a series expression for \( f \) term-by-term in order to arrive at a series expression for \( f' \), it may not be surprising that we may integrate a series term-by-term as well.

**Theorem (Term-by-Term Integration, p. 550):** Suppose that \( f(x) \) is defined by the power series \((1)\) and the radius of convergence \( R > 0 \). Then

\[
\int_0^x f(t) \, dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}, \quad \text{for all } x \text{ satisfying } -R < x < R.
\]
Example: We know that \( f(x) = (1+x)^{-1} \) has the power series representation \( \sum_{n=0}^{\infty} (-1)^n x^n \) for \( x \) in the interval \((-1, 1)\). Moreover,
\[
\int_0^x \frac{dt}{1+t} = \left[ \ln |1+t| \right]_0^x = \ln |1+x|.
\]

By the term-by-term integration theorem, for \(-1 < x < 1\) we also have
\[
\int_0^x \frac{dt}{1+t} = \int_0^x \left( \sum_{n=0}^{\infty} (-1)^n t^n \right) dt = \int_0^x (1 - t + t^2 - t^3 + t^4 - t^5 + \cdots) dt
\]
\[
= \int_0^x dt - \int_0^x t dt + \int_0^x t^2 dt - \int_0^x t^3 dt + \cdots \quad \text{(the theorem justifies this step)}
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \left( \int_0^x t^n dt \right)
\]
\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left[ t^{n+1} \right]^x_0
\]
\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}
\]
\[
= x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \cdots.
\]

That is,
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \ln(1+x),
\]
at least for all \( x \) in the interval \(-1 < x < 1\). In fact, though the theorem does not go so far as to guarantee convergence at the value \( x = 1 \), since the series on the left converges at \( x = 1 \) (Why?), one might suspect that
\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.
\]

This, indeed, is the case.

Example: Use the power series representation for \((1+x^2)^{-1}\) about zero to get a power series representation for \( \arctan x \).