Thus far in MATH 161/162, definite integrals \( \int_a^b f(x) \, dx \) have:

- been over regions of integration which were finite in length (i.e., \( a \neq -\infty \) and \( b \neq \infty \))
- involved integrands \( f \) which are finite throughout the region of integration

Q: How would we make sense of definite (improper, as they are called) integrals that violate one or both of these assumptions?

A: As limits (or sums of limits), when they exist, of definite integrals.

- When all of the limits involved exist, the integral is said to converge.
- When even one of the limits involved does not exist, the integral is said to diverge.

Examples:

\[
\int_3^\infty \frac{dx}{x^3}
\]

\[
\int_0^1 \frac{dx}{\sqrt{x}}
\]

\[
\int_0^2 \frac{dx}{(x-1)^{2/3}}
\]

\[
\int_1^\infty \frac{dx}{x \sqrt{x^2 - 1}}
\]
Evaluating them:

\[ \int_{-\infty}^{0} e^x \, dx \]

\[ \int_{0}^{1} \ln x \, dx \]

\[ \int_{1}^{\infty} \frac{dx}{x^p} \]

\[ \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} \]

Even when an improper integral cannot be evaluated exactly, one might be able to determine if it converges or not. One of several possible theorems which address this issue:

**Theorem (Direct Comparison Test):** Suppose \( f, g \) satisfy \( 0 \leq f(x) \leq g(x) \) for all \( x \geq a \). Then

(i) \( \int_{a}^{\infty} f(x) \, dx \) converges if \( \int_{a}^{\infty} g(x) \, dx \) converges.

(ii) \( \int_{a}^{\infty} g(x) \, dx \) diverges if \( \int_{a}^{\infty} f(x) \, dx \) diverges.