Math 162
Review of Series

1. (a) Explain what is meant by $\int_1^\infty f(x)\,dx$. What analogy (analogies) exists between such an improper integral and an infinite series $\sum_{n=1}^\infty a_n$?

   An improper integral with infinite interval of integration should be understood as a limit of proper integrals (in fact, that is how it is defined):

   $$\int_1^\infty \;:=\; \lim_{b\to\infty}\int_1^b f(x)\,dx.$$

   Similarly, an infinite series $\sum_n a_n$ should also be thought of as a limit of ''proper'' (or *partial*) sums. More specifically, we define the sequence of partial sums s_1, s_2, s_3, etc. by

   $$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \quad \ldots, \quad s_k = \sum_{n=1}^k a_n, \quad \ldots,$$

   then

   $$\sum_{n=1}^\infty a_n \;:=\; \lim_{k\to\infty} s_k.$$

   There is another analogy between improper integrals of the type above and infinite series, which involves areas under curves. See the second page of the Framework from Feb. 13 for more on this.

   (b) What do we mean by the phrase "$\sum a_n$ converges"? Use your definition to explain why the series $\sum_{n=1}^\infty (-1)^{n-1} = 1 - 1 + 1 - 1 + \cdots$ is considered to be divergent.

   The series $\sum_n a_n$ converges precisely when the sequence of partial sums s_1, s_2, s_3, ..., s_n, ...(defined in part (a) above) converges to a limit. Otherwise, the series diverges.

   (c) Explain how we determined which p-series converge and which ones diverge.

   A full and complete answer is given on the first page of the framework for Feb. 13. Briefly, it involves

   - the fact that improper integrals of the form $\int_1^\infty x^{-p}\,dx$ converge for $p > 1$ and diverge for $p \leq 1$, and
   - the direct comparison test (Framework for Feb. 9).

   (d) Suppose $\sum a_n$ is a convergent series and we would like to know its (exact) sum. What must one have in order to evaluate this sum as a limit? For what types of series were we able to do this? When a given series does not allow us to find its sum, what kinds of questions can still be answered concerning it? What tools are used to answer these questions?

   To get the exact sum, one needs to be able to evaluate $\lim_{n\to\infty} s_n$. But to do that requires a ''nice'' (closed-form) expression for s_n, something more easily manipulated than the basic expression $s_n = a_1 + \ldots + a_n$.
   In our explorations of series, we have been able to obtain a closed-form

expression for s_n only in the case where $\sum_n a_n$ is a geometric or telescoping series.

Even when we cannot find a series sum, we can often answer the question of whether a series converges or diverges. In that pursuit, we have several useful tests of convergence. See problem 3 for a list of the tests we have worked to learn.

2. Write down 5 made-up examples of power series (do not have them all centered at the same place). Can you determine the interval of convergence for any of those that you wrote?

   Now write down 5 made-up examples of series which are not power series. Can you determine if they converge?

   Two possible power series are

$$\sum_{n=0}^{\infty}(2x+1)^n \qquad \text{and} \qquad \sum_{n=11}^{\infty}\frac{3!}{2^n}(x-2)^{n+2},$$

   the first centered at $(-1/2)$ and the second at 2. Though both

$$\sum_{n=0}^{\infty}\sqrt{x^n} \qquad \text{and} \qquad \sum_{n=1}^{\infty}5^n(x-2)^{n-7}$$

   have an x (a true variable), the existence of fractional and negative exponents makes them not power series. (Recall that a power series is like a polynomial of infinite degree.)

   Some non-power series include these sequences involving only real numbers

$$\sum_{n=1}^{\infty}\sqrt{n} \qquad \text{and} \qquad \sum_{n=8}^{\infty}\frac{1}{n^2-105}.$$

3. Suppose we wish to know if an infinite series $\sum_n a_n$ converges. List the various tests for convergence we learned. Which test would be most reasonable to try first? Second? etc. Give reasons for why this would be a reasonable order in which to attempt their use. If your series was a power series, is there a different test you would try first?

   We have learned just three tests, the nth-term test, the ratio test and the absolute convergence test. Of these, I would tend toward using the nth-term test first when dealing with a series whose terms are real numbers (i.e., not a power series), as it is the most universally applicable. On the other hand, for a power series $\sum_n c_n(x-a)^n$, I would use the absolute convergence and ratio tests (in tandem) first (i.e., the ratio I'd be looking at is $|c_{n+1}(x-a)^{n+1}|/|c_n(x-a)^n|$) in order to get the radius of convergence.

4. **True or false.** For those statements which are false, try to rewrite them to make them true.

(a) True Consider a given series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

If the series $\sum_{n=1000}^{\infty} a_n$ converges, then so does the full series $\sum_{n=1}^{\infty} a_n$.

(b) True Consider a given series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

If the full series $\sum_{n=1}^{\infty} a_n$ diverges, then so does the series $\sum_{n=1000}^{\infty} a_n$.

(c) False It is possible for the power series $\sum c_n(x+3)^n$ to converge at $x = 2$ and diverge at $x = -5$.

(d) False If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.

(e) False The series $\sum (-1)^n n^{-p}$ converges for $p > 1$ and diverges for $p \leq 1$.

(f) True Suppose that, by use of the ratio test, you have found the power series $\sum c_n(x-a)^n$ converges for each x in the interval $(a-R, a+R)$ with $R > 0$, and diverges for $|x-a| > R$. When testing whether the series converges at an endpoint $x = a \pm R$, the ratio test will be inconclusive (and hence it will be a waste of time to try using it for this purpose).

5. There is a misuse of the nth term test which poses a strong temptation for many, and careful measures should be taken to avoid it. Describe it.

For a series $\sum_n a_n$ to converge, it is necessary that the terms a_n farther and farther out in the sum (i.e., as $n \to \infty$) to go to zero. That is, we need $\lim_{n\to\infty} a_n = 0$. This ''necessity'' should be thought of in the same way as when we make a statement like ''water is necessary for life.'' Such a statement does not imply that water is *sufficient* for life and, similarly, $\lim_n a_n = 0$ is not sufficient to conclude that $\sum_n a_n$ converges, but it is precisely this conclusion that appears quite tempting.

6. Write the Maclaurin series for e^x, $\sin x$ and $\cos x$. What intervals of convergence do these have? What do we mean when we say that these functions "are equal to their Maclaurin series"?

These Maclaurin series may be viewed on the 2nd page of the Framework for Feb. 23. Each of the three mentioned here have intervals of convergence $(-\infty, \infty)$. When we say, for instance, that $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all real x, what we are saying is that, for any real number x (fixed), the series $\sum x^n/n!$ converges to a value that is identical to e^x.

7. Recall that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

(That is, the function $(1-x)^{-1}$ has a representation inside the interval $(-1, 1)$ as a Maclaurin series.) Starting from this series, use a variety of techniques (namely, "substitution", term-by-term integration, term-by-term differentiation, multiplication by x, and combinations of these) to find "related" functions which also have series representations. Indicate, as best you can, the intervals of convergence for each.

This question is quite open-ended, intending only to have you practice various types of series manipulations. Here are examples.

Via multiplication by a power of x (or a power of $(x-a)$, if our power series were centered at a) we get

$$\frac{x^5}{1-x} = x^5 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+5}, \qquad -1 < x < 1.$$

Via substitution we get

$$\frac{1}{1+5x} = \sum_{n=0}^{\infty} (-1)^n (5x)^n, \qquad -\frac{1}{5} < x < \frac{1}{5}.$$

Via term-by-term integration we get

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}, \qquad -1 < x < 1.$$

Combining substitution and term-by-term integration, we can get that

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \qquad -1 < x < 1.$$

(Actually, this last equality holds at $x = 1$ as well, but that would be something tested separately.)

Via term-by-term differentiation, since $d/dx(1-x)^{-1} = (1-x)^{-2}$, we get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}, \qquad -1 < x < 1.$$

Note that the Maclaurin series for $(1-x)^{-1}$ is simply used as a starting point for these manipulations. One could use other power series as starting points as well. For instance, since we know

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad -\infty < x < \infty,$$

we may use substitution of $(-x^2)$ for x to get that

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}, \qquad -\infty < x < \infty.$$

In preparation for Exam 2 (the one covering series), a worksheet with a number of series was passed out. As some of you may have thrown that sheet away, the problems are duplicated below.

1. Determine whether the series converges or diverges. If convergent, say in what manner (conditionally, or absolutely), and find the series sum when possible.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

   (b) $\displaystyle\sum_{n=50}^{\infty} \frac{n^2}{(3n+1)(n-4)}$

   (c) $\displaystyle\frac{5}{\sqrt{3}} + \frac{5}{2} + \frac{5}{\sqrt{5}} + \frac{5}{\sqrt{6}} + \frac{5}{\sqrt{7}} + \cdots$

   (d) $\displaystyle 96 - 48 - 24 + 12 - 6 - 3 + \frac{3}{2} - \frac{3}{4} - \frac{3}{8} + \cdots$

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n+2)}$

   (f) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$

   (g) $\displaystyle\sum_{n=2}^{\infty} \frac{1 + (-1)^n 3^{n+2}}{5^n}$

   (h) $\displaystyle\sum_{n=10}^{\infty} \frac{(-1)^n}{n^{1/10}}$

2. Determine the radius and interval of convergence for the power series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+3)7^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!(2x-1)^n}{2n-1}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$

3. Find MacLaurin series for the following functions, and state the interval in which the your series equals the given function.

   (a) $\displaystyle\frac{7x^2}{1+9x^2}$

   (b) $f(x) := \begin{cases} \dfrac{\cos(x^2) - 1}{x^4}, & x \neq 0, \\ -\dfrac{1}{2}, & x = 0. \end{cases}$