Math 162
Review of Integration: Functions of a single variable

Evaluate the following integrals or, if they diverge, indicate this.

1. $\displaystyle\int \sin^4 x\,dx$

   We have

$$\begin{aligned}\int \sin^4 x\,dx &= \frac{1}{4}\int [1-\cos(2x)]\,dx \\ &= \frac{1}{4}\int [1-2\cos(2x)+\cos^2(2x)]\,dx \\ &= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{8}\int [1+\cos(4x)]\,dx \\ &= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.\end{aligned}$$

2. $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

   By long division, we have that

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}.$$

   Thus,

$$\int \frac{x^3+x}{x-1} = \int \left(x^2+x+2+\frac{2}{x-1}\right)dx = \frac{1}{3}x^3+\frac{1}{2}x^2+2x+2\ln|x-1|+C.$$

3. $\displaystyle\int_1^\infty \frac{\ln x}{x^{3/2}}\,dx$

   Choosing to deal with an indefinite integral for now, we have

$$\begin{aligned}\int x^{-3/2}\ln x\,dx &= -2x^{-1/2}\ln x + 2\int x^{-3/2}\,dx \quad \text{(integrating by parts)} \\ &= -2x^{-1/2}\ln x - 4x^{-1/2}.\end{aligned}$$

   Being an improper integral, we must write our original integral as a limit of proper integrals:

$$\begin{aligned}\int_1^\infty x^{-3/2}\ln x\,dx &= \lim_{b\to\infty}\int_1^b x^{3/2}\ln x\,dx \\ &= \lim_{b\to\infty}\Big[-2x^{-1/2}\ln x - 4x^{-1/2}\Big]_1^b \\ &= \lim_{b\to\infty}\left[4-\frac{2\ln b}{\sqrt{b}}-\frac{4}{\sqrt{b}}\right]\end{aligned}$$

$$\begin{aligned}
&= 4 - 2\left(\lim_{b\to\infty} \frac{\ln b}{\sqrt{b}}\right) - \lim_{b\to\infty} \frac{4}{\sqrt{b}} \\
&= 4 - 2\left(\lim_{b\to\infty} \frac{\ln b}{\sqrt{b}}\right) \\
&= 4 - 2\left(\lim_{b\to\infty} \frac{b^{-1}}{(1/2)b^{-1/2}}\right) \qquad \texttt{(by L'Hôpital's Rule)} \\
&= 4 - 2\left(\lim_{b\to\infty} \frac{2}{b^{1/2}}\right) \\
&= 4.
\end{aligned}$$

4. $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

   I will use the trig. substitution $x = 3\sin\theta$ (so $dx = 3\cos\theta\,d\theta$). Then

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\,3\cos\theta\,d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta \\
&= \int \cot^2\theta\,d\theta \\
&= \int (\csc^2\theta - 1)\,d\theta \\
&= -\cot\theta - \theta + C \\
&= -\frac{\sqrt{9-x^2}}{x} - \texttt{arcsin}\,(x/3) + C.
\end{aligned}$$

5. $\displaystyle\int \cos^3\theta\,d\theta$

   We have

$$\begin{aligned}
\int \cos^3\theta\,d\theta &= \int \cos^2\theta\cdot\cos\theta\,d\theta \\
&= \int (1-\sin^2\theta)\cos\theta\,d\theta \\
&= \int \cos\theta\,d\theta - \int \sin^2\theta\cos\theta\,d\theta \\
&= \sin\theta - \int u^2\,du \qquad \texttt{(substituting } u = \sin\theta\texttt{)} \\
&= \sin\theta - \frac{1}{3}u^3 + C \\
&= \sin\theta - \frac{1}{3}\sin^3\theta + C.
\end{aligned}$$

6. $\displaystyle\int_1^4 \frac{11}{(2x+3)^2}\,dx$

Let $u = 2x+3$, so $du = 2\,dx$. Then

$$\begin{aligned}
\int_1^4 \frac{11}{(2x+3)^2}\,dx &= \frac{11}{2}\int_5^{11} u^{-2}\,du \\
&= -\frac{11}{2}\left[\frac{1}{u}\right]_5^{11} \\
&= -\frac{11}{2}\left(\frac{1}{11}-\frac{1}{5}\right) \\
&= \frac{3}{5}.
\end{aligned}$$

7. $\displaystyle\int_0^2 \frac{dx}{5x-1}$

The integrand $(5x-1)^{-1}$ has an infinite discontinuity at $x = 1/5$, which is between the limits of integration. Thus, this integral is improper and is rightfully expressed as the sum of limits

$$\int_0^2 \frac{dx}{5x-1} = \lim_{a\to(1/5)^-}\int_0^a \frac{dx}{5x-1} + \lim_{b\to(1/5)^+}\int_b^2 \frac{dx}{5x-1}.$$

But neither of these limits exists. For instance,

$$\begin{aligned}
\lim_{b\to(1/5)^+}\int_b^2 \frac{dx}{5x-1} &= \lim_{b\to(1/5)^+}\frac{1}{5}\ln|5x-1|\Big]_b^2 \\
&= \frac{1}{5}\left[\ln 9 - \left(\lim_{b\to(1/5)^+}\ln(5b-1)\right)\right] \\
&= +\infty.
\end{aligned}$$

Since one (and even both) of the limits does not exist, the original integral diverges.

8. $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$

The integrand here points to another trig. substitution, this time $x = (3/2)\tan\theta$ (so $dx = (3/2)\sec^2\theta\,d\theta$). This gives us

$$\begin{aligned}
\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx &= \int_0^{\pi/3} \frac{(3/2)^3\tan^3\theta\cdot(3/2)\sec^2\theta}{[4(3/2)^2\tan^2\theta+9]^{3/2}}\,d\theta \\
&= \left(\frac{3}{2}\right)^4\int_0^{\pi/3}\frac{\tan^3\theta\sec^2\theta}{[9\tan^2\theta+9]^{3/2}}\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3}\frac{\tan^3\theta\sec^2\theta}{(1+\tan^2\theta)^{3/2}}\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta\sec^2\theta}{\sec^3\theta}\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{\tan\theta(\sec^2\theta-1)}{\sec\theta}\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \left(\sec\theta\tan\theta\,d\theta - \frac{\tan\theta}{\sec\theta}\right)\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \left(\sec\theta\tan\theta\,d\theta - \sin\theta\right)\,d\theta \\
&= \frac{3}{16}\Big[\sec\theta + \cos\theta\Big]_0^{pi/3} \\
&= \frac{3}{16}\left(2 + \frac{1}{2} - 2\right) \\
&= \frac{3}{32}.
\end{aligned}
$$

9. $\displaystyle\int \frac{2x+1}{x^2-3x+2}\,dx$

The integrand is a rational function, whose denominator factors to $(x-2)(x-1)$. So, we employ partial fraction expansion:

$$
\frac{2x+1}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1} \quad\Rightarrow\quad 2x+1 = A(x-1) + B(x-2)
$$
$$
\Rightarrow\quad A = 5,\ \ B = -3.
$$

So,

$$
\begin{aligned}
\int \frac{2x+1}{x^2-3x+2}\,dx &= 5\int \frac{dx}{x-2} - 3\int\frac{dx}{x-1} \\
&= 5\ln|x-2| - 3\ln|x-1| + C.
\end{aligned}
$$