MATH 162: Calculus II, Sections B/C

Information for Exam 3
to take place on Apr. 5, 2007

General Information. The exam focuses on material covered in frameworks for the dates Feb. 27–Mar. 30. Problems during this time have been assigned from sections

10.1, 10.5, 10.6: graphs in space
10.2–10.4: vectors
11.1–11.2: vector functions
12.1–12.6: functions of multiple variables

There will be no formulas provided on the exam, and it is expected that the only memory to which you have committed formulas is the biology-based memory using your own brain cells. Some useful formulas you might look at closely include

1. $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$, distance between points (x_0, y_0, z_0) and (x_1, y_1, z_1)
2. $\mathbf{u} \cdot \mathbf{v} \;=\; |\mathbf{u}||\mathbf{v}| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$
3. scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$: $\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$
4. $\text{proj}_{\mathbf{v}}\mathbf{u} \;=\; \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}$
5. $\mathbf{u} \times \mathbf{v} \;=\; (|\mathbf{u}||\mathbf{v}| \sin\theta)\mathbf{n} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$,

   where $\mathbf{n}$ is a unit vector so that $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ form a right-hand system
6. distance from S to line parallel to $\mathbf{v}$ containing P: $\dfrac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$
7. distance from S to plane normal to $\mathbf{n}$ containing P: $\dfrac{|\overrightarrow{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}$
8. $f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$,

   giving the equation of tangent plane to $f(x, y, z) = f(z_0, y_0, z_0)$ at (x_0, y_0, z_0)
9. $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$,

   giving the equation of tangent plane to $z = f(x, y)$ at (x_0, y_0)
10. $D_{\mathbf{u}}f(x, y) \;=\; \vec{\nabla} f(x, y) \cdot \mathbf{u}$, derivative of f at (x, y) in direction $\mathbf{u}$

Answers to the Incidental Questions (please alert me if you find any errors!)

1. Concepts and terms:

   (a) What you plug into such a function f are *points*, or coordinate pairs, from $\mathbb{R}^2$, not real numbers. So, the domain is accurately described as "the set of points (x, y) for which $y \neq 1$."

   (b) Some include
   $$x^2 - xy^4 + 7, \qquad \sec(1 - xy), \qquad \text{and} \qquad \frac{2}{\sqrt{x-y}}.$$
   Most anything you write down, with the exception of piecewise-defined functions, will do. An example of a function that is discontinuous along the x-axis (which is, nevertheless, part of its domain) is
   $$f(x, y) = \begin{cases} 2, & \text{if } y > 0, \\ 1, & \text{if } y = 0, \\ 0, & \text{if } y < 0. \end{cases}$$

   (c) The actual definition appears about half-way down the 2nd page on the framework for Mar. 6. A good working definition (one that is implied by the technical one you see there) is that a function is differentiable at points where, locally, it behaves like a plane (the tangent plane).

   As for how you can tell if a function is differentiable, see the first theorem on the 3rd page of that same framework.

   (d) $\mathbf{u}(t)$ is continuous (resp. differentiable) at t_0 if each of its component functions $x(t)$, $y(t)$ and $z(t)$ are continuous (resp. differentiable) there.

   (e) We mean the acute angle between the normal lines to these planes.

   (f) The upper half plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ is an open region; so is any open disk. The set $\{(x, y) \mid y \geq 0\}$ is closed. Any circle consists of only boundary points; so does the set of points making up the x-axis. The set of points $\{(x, y) \mid x \geq 0, y > 0\}$ (i.e., those in the 1st quadrant with the boundary points on the y-axis included but those on the x-axis not included) is neither open nor closed.

   (g) We mean 2nd (or higher-order) derivatives that include differentiation with respect to two (or more) variables, such as $\partial/\partial x(f_y) = f_{yx}$, etc.

   (h) Let $g = f_x$, so that $g_{xy} = f_{xxy}$ and $g_{yx} = f_{xyx}$. To be certain that $g_{xy} = g_{yx}$ at a point, we need to be able to invoke the theorem on the 1st page of the framework for Mar. 6, which means g, g_x, g_y, g_{xy} and g_{yx} must be defined in at least some disk surrounding the point in question with all being continuous at the point in question. Now rewrite g and these other derivatives in terms of f.

   (i) A quadric surface is a level set (level surface) of a quadratic polynomial in x, y and z.

2. Routine problems

   (a) $\left\langle \frac{2}{3\sqrt{6}}, -\frac{7}{3\sqrt{6}}, \frac{1}{3\sqrt{6}} \right\rangle$

   (b) i. $\left\langle \frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, 0 \right\rangle$

ii. $\mathbf{R}(t) = (s - s^2)\mathbf{i} + \left(\frac{9}{2} + \frac{3}{2}s^2 - 5s\right)\mathbf{j} + (s-3)\mathbf{k}$.

Since the curve described by $\mathbf{r}(t)$ is a line, one might guess the curve described by $\mathbf{R}(t)$ is a parabolic arc in space. This is, in fact, correct.

(c) Let $f(x, y) = x^2 - 2x - y^2 + 4y - 2$.

i. $\frac{1}{\sqrt{5}} \langle 2, 1\rangle$

ii. $4x + 2y - z = 10$

iii. $4/5$

iv. The only such point is $(1, 2)$. At this point, the tangent plane is horizontal (parallel to the xy-plane).

v. You will not be held responsible for this or other problems marked "harder". The correct answer is an hyperbola opening to the left and to the right, with the point $(1, 2)$ precisely between the two halves.

vi. The graph is an hyperbolic paraboloid (saddle), with the (central) "saddle point" located at $(1, 2, f(1, 2))$. The orientation of this hyperbolic paraboloid is such that slices parallel to the xy-plane produce hyperbolas, while slices parallel to the other planes produce parabolas.

vii. $\frac{\partial f}{\partial u} = [2u\sin(uv) - 2][\sin(uv) + uv\cos(uv)] + \left(4 - \frac{2v}{v-u}\right)\frac{v}{(v-u)^2}$

(d) Let $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

i. $\mathbf{r}(t) = (5 + 3t)\mathbf{i} - (1 + 2t)\mathbf{j} - (3 + t)\mathbf{k}$

ii. $\mathbf{r}(t) = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k} + \frac{1}{\sqrt{14}}t\mathbf{v}$

iii. One possible answer: $\mathbf{r}(t) = (5 + 3e^t)\mathbf{i} - (1 + 2e^t)\mathbf{j} - (3 + e^t)\mathbf{k}$

iv. $2\sqrt{\frac{13}{7}}$

v. $\mathbf{r}(t) = (5 - 4t)\mathbf{i} + (-1 + 2t)\mathbf{j} + (-3 + 4t)\mathbf{k}$, for $0 \le t \le 1$.

vi. $3(x - 1) - 2y - (z - 1) = 0$, or $3x - 2y - z = 2$

vii. $2/\sqrt{14}$

viii. 1.9465 (radians)

ix. scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$: $-\sqrt{7/2}$
$\text{proj}_{\mathbf{v}}\mathbf{u} = -(1/2)\mathbf{v}$

x. Technically, since the angle from 2 parts ago was obtuse, not acute, there is no pair of planes for which the angle between them would be considered to be 1.9465 radians. However, two planes can make the angle $\pi - 1.9465 = 1.1951$ radians, and, in fact, any two planes with normal vectors $\mathbf{u}$ and $\mathbf{v}$ will do. For instance, $x + 5y = 7$ and $3x - 2y - z = 0$.

xi. 0.56394 (radians)

xii. $\frac{1}{3\sqrt{35}}(-5\mathbf{i} + \mathbf{j} - 17\mathbf{k})$