1 Notation and Definitions

monomial in $x_1, \ldots, x_n$

A product of the form $x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n}$ with each $a_i \geq 0$ an integer. Writing $\alpha = (\alpha_1, \ldots, \alpha_n)$, we may define $x^\alpha := x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n}$.

$n$-dimensional affine space $k^n$ over the field $k$

$k^n := \{(a_1, \ldots, a_n) | \text{each } a_j \in k\}$

degree of monomial $x^\alpha$

The quantity $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

polynomial $f$ in $x_1, \ldots, x_n$ with coefficients in a field $k$

A finite linear combination $f = \sum a_\alpha x^\alpha$, each $a_\alpha \in k$. Note that such an $f$ gives rise to a mapping $f: k^n \to k$.

polynomial ring $k[x_1, \ldots, x_n]$

Collection of (all) polynomials in $x_1, \ldots, x_n$.

equality of polynomials $f = g$

Polynomials are formal constructions specified by their coefficients. When $f = \sum a_\alpha x^\alpha$ and $g = \sum b_\alpha x^\alpha$, the statement $f = g$ means $a_\alpha = b_\alpha$ for each $\alpha$. In particular, a polynomial $f$ is the zero polynomial, or $f = 0$, precisely when all its coefficients $a_\alpha = 0$.

equality of functions $f, g$ on affine space $k^n$:

Two functions $f, g: k^n \to k$ are equal as functions precisely when $f(b_1, \ldots, b_n) = g(b_1, \ldots, b_n)$ for every $(b_1, \ldots, b_n) \in k^n$. In particular, the zero function $f$ is the one for which $f(b_1, \ldots, b_n) = 0$ for every $(b_1, \ldots, b_n) \in k^n$.

Finite fields

For $p$ a prime, we denote the field $\mathbb{Z}_p$ instead by $\mathbb{F}_p$.

Affine varieties

Let $f_1, \ldots, f_s$ be polynomials in $k[x_1, \ldots, x_n]$. We call the set of common zeros

$V(f_1, \ldots, f_s) := \{(b_1, \ldots, b_n) \in k^n | f_i(b_1, \ldots, b_n) = 0 \text{ for each } 1 \leq i \leq s\}$

the affine variety defined by $f_1, \ldots, f_s$. 
2 Results

**Theorem 2.1 (Polynomial Vanishing Theorem).** Let $k$ be an infinite field (i.e., have infinity many distinct elements), and $f \in k[x_1, \ldots, x_n]$. Then $f = 0$ if and only if $f : k^n \to k$ is the zero function.

Remark: Note that the assumption of an infinite field is important here. The polynomial $f(x) := (x + x^2) \in \mathbb{F}_2[x]$ is not the zero polynomial. However, for each of the elements in of $\mathbb{F}_2 = \{0, 1\}$, $f(x) = 0$, making it the zero function.

Proof. It is obvious that the zero polynomial is the zero function. So, suppose $f$ is a polynomial in $k[x_1, \ldots, x_n]$ for which $f : k^n \to k$ is the zero function. We prove our claim by induction.

In the case $n = 1$, assume $f$ is of degree $m$. Were $f$ not the zero polynomial then it would have at most $m$ distinct roots in $k$. Since $k$ is infinite, there would be $b \in k$ for which $f(b) \neq 0$, which cannot be since $f$ is the zero function. Thus, $f$ is the zero polynomial.

Assume now that our proposition holds for $n - 1$. By collecting the various powers of $x_n$, we can write $f$ in the form

$$f = \sum_{i=0}^{N} g_i(x_1, \ldots, x_{n-1})x_n^i,$$

where each $g_i \in k[x_1, \ldots, x_{n-1}]$. Now, for a point $(b_1, \ldots, b_{n-1}) \in k^{n-1}$ (fixed),

$$h(x_n) := f(b_1, \ldots, b_{n-1}, x_n) = \sum_{i=0}^{N} g_i(b_1, \ldots, b_{n-1})x_n^i$$

lives in $k[x_n]$. Since $f$ is the zero function on $k^n$, $h$ is the corresponding zero function on $k$.

Our proof for the case $n = 1$ now shows that $h$ is the zero polynomial in $k[x_n]$, meaning that each of its coefficients $g_i(b_1, \ldots, b_{n-1}) = 0$. Since the choice of $(b_1, \ldots, b_{n-1})$ was arbitrary, each $g_i$ is the zero function on $k^{n-1}$ and, by the induction hypothesis, each $g_i$ is the zero polynomial in $k[x_1, \ldots, x_{n-1}]$. This completes the proof. \qed

2.1 Some exercises from the previous class

1.1.2 a. Consider the polynomial $g(x, y) = x^2y + y^2x \in \mathbb{F}_2[x, y]$. Show that $g(x, y) = 0$ for every $(x, y) \in \mathbb{F}_2^2$, and explain why this does not contradict the Polynomial Vanishing Theorem.
b. Find a nonzero polynomial in $\mathbb{F}_2[x, y, z]$ which vanishes at every point of $\mathbb{F}_2^3$. Try to find one involving all three variables.

c. Find a nonzero polynomial in $\mathbb{F}_2[x_1, \ldots, x_n]$ which vanishes at every point of $\mathbb{F}_2^n$. Can you find one in which all of $x_1, \ldots, x_n$ appear?

**Answer:**

a. The first task is to demonstrate $g(0, 0) = 0 = g(1, 0) = g(0, 1) = g(1, 1) = 0$, which is straightforward using the arithmetic of $\mathbb{F}_2$. But here is some Sage code that does it.

```python
F2 = Integers(2)
F2 # says that it is a ring, but it is really a field
AffineSpace(2, F2)
var('x y')
f(x,y) = x^2*y + y^2*x
for elem in AffineSpace(2, F2):
    pt = vector(elem)
    print f(*pt) # or, if necessary, print Z2(f(*pt))
```

Since $\mathbb{F}_2$ is a finite field, there is no violation of the Polynomial Vanishing Theorem.

b. Similar Sage commands demonstrate that

$$f(x, y, z) := x^2(y + z) + y^2(x + z) + z^2(x + y)$$

does the trick.

c. Following on the heals of my last proposal, we take

$$f(x_1, \ldots, x_n) := \sum_{j=1}^{n} x_j^2 \left( \sum_{i \neq j} x_i \right).$$

1.1.5 In the proof of the Polynomial Vanishing Theorem, we took $f \in k[x_1, \ldots, x_n]$ and wrote it as a polynomial in $x_n$ with coefficients in $k[x_1, \ldots, x_{n-1}]$. To see what this looks like in a specific case, consider the polynomial

$$f(x, y, z) = x^5y^2z - x^4y^3 + y^5 + x^2z - y^3z + xy + 2x - 5z + 3.$$

a. Write $f$ as a polynomial in $x$ with coefficients in $k[y, z]$.

b. Write $f$ as a polynomial in $y$ with coefficients in $k[x, z]$.

c. Write $f$ as a polynomial in $z$ with coefficients in $k[x, y]$.

**Answer:**

a. $f(x, y, z) = (3 - 5z - y^3z + y^5) + (2 + y)x + zx^2 - y^3x^4 + y^2zx^5$
b. $f(x, y, z) = (3 - 5z + 2x + x^2z) + xy + x^5y^2 - (x^4 + z)y^3 + y^5$
c. $f(x, y, z) = (3 + 2x + xy - x^4y^3 + y^5) + (x^2 + x^5y^2 - 5 - y^3)z$
3 Sage

3.1 Some basic constructs

```python
a = [1, 2, 3]  # a list
type(a)

b = (4, 5, 6)  # a tuple
type(a)

a + b  # produces error

c = range(2, 5)  # a list
a + c  # probably not what class expects

vector(a) + vector(b)

u = vector(a)
v = vector(a)
plot(u) + plot(v) + plot(u+v, color='red', thickness=2)

for elem in a:  # colon is required
    b = elemˆ2
    b  # loop ends when no longer indenting
```

3.2 Defining functions and plotting them: 2-dimensional case

```python
var('x')
f(x) = sin(x) / x
plot(f(x), (x, -6*pi, 6*pi))

var('t')
x(t) = 3*cos(t)
y(t) = 4*sin(t)
plot(x(t), (t, -pi, pi))
plot(y(t), (t, -pi, pi))
parametric_plot((x(t),y(t)), (t, -pi, pi), aspect_ratio=1)
```

The last ellipse may be expressed without reference to a parameter \( t \), but rather as the polynomial equation in \( x \) and \( y \):

\[
\frac{x^2}{9} + \frac{y^2}{16} = 1.
\]

Of course, there is a relationship expressed here between \( x \) and \( y \), but this relationship does not constitute a (single) function. Rather, the relation implicitly defines two functions:

\[
y = 4 \sqrt{1 - \frac{x^2}{9}} \quad \text{and} \quad y = -4 \sqrt{1 - \frac{x^2}{9}},
\]

but the entire thing may be plotted without these algebraic manipulations via
3.3 Plotting in 3 dimensions

Some equations in the variables $x, y, z$ may be solved explicitly for $z$. In such cases, plots of their graphs are produced in a similar manner to their corresponding 2-variable counterparts. For instance, consider the equation

$$x^2 - 2xy + z = 11,$$

whose graph, a surface, consists of the points $(x, y, z)$ for which $z = 11 + 2xy - x^2$.

Once again, however, there is the option of obtaining the plot without the effort (easy here, but is not always) of solving for $z$:

```python
f(x, y) = 11 + 2*x*y - x^2
plot3d(f(x, y), (x, -5, 5), (y, -5, 5))
```

Questions:

1. Consider the circle of radius 1 that is parallel to the unit circle but lies in the plane $z = 2$—that is, the circle described by the pair of equations

$$x^2 + y^2 = 1, \quad z = 2.$$  

How might such a circle be plotted in 3 dimensions. [Hint: Use the `implicit_plot3d()` command.]

2. How might our surface $z = 11 + 2xy - x^2$ be plotted using the `implicit_plot3d()` command? [Note: This is a new kind of object—a surface instead of a curve—we are plotting with `implicit_plot3d()`. A curve requires only one parameter, but a surface requires two.]

3. Considering that the conversion from spherical coordinates to rectangular ones is given by

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi,$$

what command involving `implicit_plot3d()` would produce the part of the unit sphere that resides in the first octant?
4 Affine Varieties

Our last plot, however we obtain it, is the affine variety (see definition above) \( V(f(x, y, z)) \), where \( f(x, y, z) = x^2 - 2xy + z - 11 \in \mathbb{R}[x, y, z] \). Some other examples:

1. The ellipse of points in \( \mathbb{R}^2 \) satisfying

\[
\frac{x^2}{9} + \frac{y^2}{16} = 1
\]

is an affine variety, given by the notation \( V(x^2/9 + y^2/16 - 1) \).

The hyperbola given by

\[
\frac{x^2}{9} - \frac{y^2}{16} = 1
\]

is an affine variety as well.

In fact, all conic sections are affine varieties defined by a single 2nd-degree polynomial in \( x \) and \( y \).

Note that, here, \( s = 1 \) and \( n = 2 \), and the resulting varieties are curves (1-dimensional objects).

2. Given any (single-variable) polynomial \( p \), the graph of the function \( y = p(x) \) is the affine variety \( V(y - p(x)) \).

Given any (two-variable) polynomial \( p \), the graph of the function \( z = p(x, y) \) is the affine variety \( V(z - p(x, y)) \).

3. Any quadric surface consisting of points in \( \mathbb{R}^3 \) that satisfy

\[
Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 + Gx + Hy + Iz + J = 0
\]

is an affine variety. Display some of

- \( 9x^2 - y^2 - 4z = 0 \) (hyperbolic paraboloid)
- \( 9x^2 + y^2 - 4z^2 = 0 \) (elliptic cone)
- \( 9x^2 + y^2 - 4z^2 = 1 \) (hyperboloid in one sheet)
- \( 9x^2 - y^2 - 4z^2 = 1 \) (hyperboloid in two sheets)
- \( 9x^2 + y^2 + 4z^2 = 1 \) (ellipsoid)

Note that, in these cases where \( s = 1 \) and \( n = 3 \), the resulting varieties are surfaces (2-dimensional objects).
4. **Twisted Cubic** (see pp. 7–8 in text)

Consider the affine variety consisting of points in \( \mathbb{R}^3 \) \( V(y - x^2, z - x^3) \). Note that, speaking generally,

\[
V(f_1, \ldots, f_s) = V(f_1) \cap V(f_2) \cap \cdots \cap V(f_s)
\]

so we seek the intersection of \( V(y - x^2) \) (its graph plotted in \( \mathbb{R}^3 \)) and \( V(z - x^3) \).

```python
p1 = implicit_plot3d(y-x^2==0, (x,-2,2), (y,-2,2), (z,-2,2), color='blue', opacity=.7)
p2 = implicit_plot3d(z-x^3==0, (x,-2,2), (y,-2,2), (z,-2,2), color='red', opacity=.7)
show(p1 + p2)
```

In our example here, \( s = 2 \) and \( n = 3 \), with the resulting variety again being a curve (a 1-dimensional object).

5. Though our intuition may tell us the dimension of an affine variety is \( (n - s) \), this is not always the case, as the example \( V(xz, yz) \) demonstrates. This affine variety is not even the union of surfaces, but rather of the \( xy \)-plane (surface) and the \( z \)-axis (line).

6. Consider a matrix problem \( Ax = b \), where \( A = (a_{ij}) \) is \( m \)-by-\( n \), with \( m < n \). The solution(s) are points \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) which simultaneously satisfy the \( m \) equations

\[
\begin{align*}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1, \\
\vdots & \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m.
\end{align*}
\]

That is, the solutions are precisely those points in the variety \( V(f_1, \ldots, f_m) \), where the \( f_i, 1 \leq i \leq m \), are defined as

\[
f_i(x_1, \ldots, x_n) := a_{i1}x_1 + \cdots + a_{in}x_n - b_i \in \mathbb{R}[x_1, \ldots, x_n].
\]

As matrix multiplication is a linear operation, such affine varieties will be called **linear varieties**.

Note, the dimension of a linear variety is not automatically \( n - m \). As an example, the linear variety \( V(x + 2y + 3z - 1) \) is just the solutions of the linear equation

\[
x + 2y + 3z = \begin{bmatrix} 1 & 2 & 3 \\ x \\ y \\ z \end{bmatrix} = 1,
\]

a **plane** (dimension = \( 3 - 1 = 2 \)). The variety \( V(x + 2y + 3z - 1, 2x + 4y + 6z - 2) \) consists of the solutions to the system of linear equations

\[
x + 2y + 3z = 1 \\
2x + 4y + 6z = 2
\]
which is clearly the same set as the variety above. On the other hand, $V(x + 2y + 3z - 1, 2x + 4y + 6z)$ consists of the solutions, of which there are none, to the system of linear equations

\[
\begin{align*}
x + 2y + 3z &= 1 \\
2x + 4y + 6z &= 0
\end{align*}
\]

The former shows we need some notion of the number $r$ of independent equations (in the matrix-problem case, rank($A$), gives this), in which case we might expect the dimension of the variety to be $(n - r)$. The latter case shows the need, even equipped with $r$, for some type of compatibility between the polynomials defining the variety before we could draw any conclusion about dimension.

7. In multivariable calculus, some may have encountered both the idea of a constrained optimization problem, where we seek to maximize/minimize an objective function $f(x, y, z)$ subject to some constraint $g(x, y, z) = 0$, and the Lagrange multiplier method for solving such a problem, which says to look for such extrema at points where $\nabla f = \lambda \nabla g$ (i.e., points at which the gradients of the two functions are parallel).

As an example, suppose we seek to find the point on the sphere $x^2 + y^2 + z^2 = 4$ which is farthest from the point $(1, -1, 1)$. Here, our objective function—the one to maximize—might be the distance function

\[
\sqrt{(x - 1)^2 + (y + 1)^2 + (z - 1)^2}.
\]

(Note: We are maximizing under the constraint that our point must be on the sphere; without any constraint, there is no maximum distance.) However, since that is not a polynomial (i.e., not in $\mathbb{R}[x, y, z]$), and since maximizing the square of the distance would have the same effect, we take our objective function to be

\[
f(x, y, z) = (x - 1)^2 + (y + 1)^2 + (z - 1)^2,
\]

and our constraint function to be

\[
g(x, y, z) = x^2 + y^2 + z^2 - 4.
\]

To the constraint equation $g(x, y, z) = 0$ we add the three (latter) equations arising from $\nabla f - \lambda \nabla g = 0$:

\[
\begin{align*}
f_1(x, y, z, \lambda) &:= x^2 + y^2 + z^2 - 4 = 0, \\
f_2(x, y, z, \lambda) &:= 2(x - 1) - 2\lambda x = 0, \\
f_3(x, y, z, \lambda) &:= 2(y - 1) - 2\lambda y = 0, \\
f_4(x, y, z, \lambda) &:= 2(z - 1) - 2\lambda z = 0.
\end{align*}
\]
Note that each $f_i \in \mathbb{R}[x, y, z, \lambda]$, and the solutions form a variety which, hopefully, will be of dimension $4 - 4 = 0$ (i.e., distinct points).

**Assignment:** In Section 1.2 do Exercises 1c, 2, 4cf, and 5. Read Exercise 8, which says that the set $X = \{(x, x) | x \in \mathbb{R}, x \neq 1\}$ is not an affine variety in $\mathbb{R}^2$. Show that the set $\overline{X} := \{(x, x) | x \in \mathbb{R}\}$ is an affine variety. In what sense(s) might we say that $\overline{X}$ is the *smallest* affine variety containing $X$?
Lemma 4.1. Suppose $V, W \subset k^n$ are affine varieties. Then $V \cap W$ and $V \cup W$ are affine varieties in $k^n$ as well.

Proof. By definition, there are polynomials $f_i, g_j \in k[x_1, \ldots, x_n]$ for $1 \leq i \leq s, 1 \leq j \leq t$ for which $V = V(f_1, \ldots, f_s)$ and $W = W(g_1, \ldots, g_t)$. As discussed already,

$$V \cap W = V(f_1, \ldots, f_s, g_1, \ldots, g_t),$$

showing the former is an affine variety.

We show that $V \cup W$ is an affine variety by showing that

$$V \cup W = V(f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t).$$

Notice that each $f_i g_j$ is a polynomial in $k[x_1, \ldots, x_n]$, so the right-hand side is truly a variety, which we'll label $U$.

Suppose a point $(b_1, \ldots, b_n)$ is in $V \cup W$. This point must either be in $V$ or $W$, let's say $V$. Then $f_i(b_1, \ldots, b_n) = 0$ for each $1 \leq i \leq s$, and hence $(f_i g_j)(b_1, \ldots, b_n) = 0$ for each $1 \leq i \leq s, 1 \leq j \leq t$. That is, we have demonstrated the inclusion "$(V \cup W) \subset U$".

To get the other inclusion, suppose $(b_1, \ldots, b_n) \in U$. Either this point of $k^n$ lies in $V$ or it does not. Assuming it does not, there is some $i_0 \in \{1, \ldots, s\}$ for which $f_{i_0}(b_1, \ldots, b_n) \neq 0$. If there were also some $j_0 \in \{1, \ldots, t\}$ for which $g_{j_0}(b_1, \ldots, b_n) \neq 0$, the it would be the case that $(f_{i_0} g_{j_0})(b_1, \ldots, b_n) \neq 0$, and $(b_1, \ldots, b_n)$ would not be in $U$. Thus, $g_j(b_1, \ldots, b_n) = 0$ for each $1 \leq j \leq t$, and $(b_1, \ldots, b_n) \in W$. Thus, each point in $U$ lies in one of $V$ or $W$, and hence $V \cup W$. This completes the proof. \qed

Note: It is possible (see homework) to use induction to get that finite unions and intersections of affine varieties again are affine varieties. In fact, the intersection of an arbitrary (even uncountably infinite) collection of affine varieties is, again, an affine variety. But there are examples of infinite unions of varities which are not, themselves, varieties. By these observations, we say that, given some set $A \subset k^n$, if we define $C$ to be the collection of all affine varieties in $k^n$ containing $A$ (a nonempty set, since $k^n$ is, itself, a variety), then $\cap_{V \in C} V$ is the smallest affine variety containing $A$.

Assignment: In Section 1.2 do Exercises 7 and 15a.
5 Parametrizing Affine Varieties

5.1 Some definitions

Rational function in $t_1, \ldots, t_m$ with coefficients in $k$

A quotient $f/g$ of polynomials $f, g \in k[t_1, \ldots, t_m]$, with the requirement that $g$ is not the zero polynomial. The collection of all such rational functions, a set which satisfies the field properties under the usual definitions of addition and multiplication, is denoted by $k(t_1, \ldots, t_m)$.

Equality of rational functions

Two rational functions in $f/g, h/k$ in $k(t_1, \ldots, t_m)$ are equal when the polynomials $kf$ and $gh$ in $k[t_1, \ldots, t_m]$ are equal.

Rational parametric representation of an affine variety $V$

Let $V$ be an affine variety in $k^n$, and $r_1, \ldots, r_n \in k(t_1, \ldots, t_m)$ such that

1. the points given by

\[
\begin{align*}
x_1 &= r_1(t_1, \ldots, t_m), \\
x_2 &= r_2(t_1, \ldots, t_m), \\
&\vdots \\
x_n &= r_n(t_1, \ldots, t_m)
\end{align*}
\]

all lie in $V$, and

2. $V$ is the smallest variety in $k^n$ containing all the image points of these rational functions.

Then these rational functions comprise a rational parametric representation of $V$. Note: It is a special affine variety, a type known as a unirational affine variety, which has such a parametrization; most do not. When each of $r_1, \ldots, r_m$ is, in fact, a polynomial in $k[t_1, \ldots, t_m]$ we say we have a polynomial parametric representation of $V$.

Implicit representation of an affine variety

For $V(f_1, \ldots, f_s)$ with each $f_j \in k[x_1, \ldots, x_n]$, the defining equations $f_1 = \cdots = f_s = 0$ are called an implicit representation of the affine variety.
5.2 Some parametric representations

For lines in \( \mathbb{R}^3 \):

You might know the line you want as “the one which is the intersection of these two planes”—i.e., the collection of points which simultaneously solve the equations

\[
\begin{align*}
A_1x + B_1y + C_1z - D_1 &= 0, \\
A_2x + B_2y + C_2z - D_2 &= 0.
\end{align*}
\]

In this case you are beginning with an implicit representation for your affine (linear) variety (i.e., the line in question). We’ll say nothing here about this case, but see below.

Other common ways to specify a line:

- Provide two points through which the line passes, or
- Give one point through which the line passes along with a direction vector.

As the latter is obtainable from the former, we’ll handle only that case. Let \( P(x_0, y_0, z_0) \) be the specified point, and \( v = (v_1, v_2, v_3) \neq 0 \) be a “direction” vector (in practice, we need not require \( \|v\| = 1 \)). For any point \( Q(x, y, z) \) on the line, the vector \( \overrightarrow{PQ} \) is a scalar multiple of \( v \). Specify that scalar—we’ll call it \( t \), which will serve as the parameter—and you have specified the point \( Q \). Thus, we have \( \overrightarrow{PQ} = tv \) or, written out in the form (1) (as a polynomial parametric representation, with polynomials in \( \mathbb{R}[t] \)), we have

\[
\begin{align*}
x &= x_0 + tv_1, \\
y &= y_0 + tv_2, \\
z &= z_0 + tv_3.
\end{align*}
\]

Note: It is not very difficult to generalize these ideas to the parametrization of an arbitrary line in \( \mathbb{R}^n \) or \( \mathbb{C}^n \).

If using \texttt{SAGE} to plot the line through \((1, -1, 2)\) in the direction \((-3, 1, 1)\), the above discussion means you could do so via a command like

\[
\text{parametric_plot3d}((1 - 3*t, -1 + t, 2 + t), (t, -2, 2))
\]

(You might experiment with various alternatives to the \( t \)-interval \(-2 \leq t \leq 2 \).)

For planes in \( \mathbb{R}^3 \):
A plane is often specified in the implicit form \( Ax + By + Cz = D \), where the constants \( A, B, C \) and \( D \) are known. We might view this from the standpoint that each of the variables \( x, y, z \) introduces a degree of freedom, while each independent constraint (here there is only one) removes one. If we let \( y, z \) remain free (permissible so long as \( A \neq 0 \), otherwise we would choose a different pair of variables to be free) and assign them as parameters \( y = s, z = t \) (the change in name isn’t really necessary), then the constraint finishes off the (polynomial) parametrization in \( \mathbb{R}[s,t] \)

\[
\begin{align*}
x &= \frac{1}{A}(D - Bs - Ct), \\
y &= s, \\
z &= t.
\end{align*}
\]

Of course, another way to write the implicit form is as a matrix equation

\[
\begin{bmatrix}
A & B & C
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = D.
\]

General linear varieties specified via matrix equations are handled next, and the parametrization of a plane in \( \mathbb{R}^3 \) should probably be thought of as a special instance of the approach demonstrated below.

Suppose you wanted to plot the plane \( 3x + y - 2z = 7 \) using \texttt{SAGE}. Following the previous discussion, you could do so with a command like

```plaintext
parametric_plot3d((7 - s + 2*t, s, t), (s, -5, 5), (t, -5, 5))
```

Again, it might be useful to play around with the \( s \) and \( t \)-ranges.

For linear varieties in \( \mathbb{R}^n \):

Note, first, that each of the previous two cases are special cases of linear varieties in Euclidean space. We assume that we start with the defining equations (i.e., an implicit representation) of the variety \( V \) in question—that is, the constants \( a_{ij}, b_i, 1 \leq i \leq m, 1 \leq j \leq n \) in the equations below are all known to us.

\[
\begin{align*}
f_1(x_1, \ldots, x_n) &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1 = 0, \\
& \vdots \\
f_m(x_1, \ldots, x_n) &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - b_m = 0.
\end{align*}
\]

Equivalently, we have the matrix \( A = (a_{ij}) \) and right-hand side vector \( b = (b_1, \ldots, b_m) \) in the matrix equation \( Ax = b \).
In this case we may obtain a parametrization of the linear variety by solving—likely using Gaussian elimination—the matrix equation. As an example, consider the matrix equation

\[
\begin{bmatrix}
2 & 3 & 0 & -1 \\
1 & 0 & 3 & 1 \\
-3 & -5 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
-1 \\
7 \\
4
\end{bmatrix},
\]

whose solutions comprise a linear variety in \(\mathbb{R}^4\). (We do not know this variety is anything other than the empty set, but solving the problem will reveal this if it is the case.) Performing elementary operations on the augmented matrix, one might arrive at an echelon form (the one on the right)

\[
\begin{bmatrix}
2 & 3 & 0 & -1 & -1 \\
1 & 0 & 3 & 1 & 7 \\
-3 & -5 & 1 & 2 & 4
\end{bmatrix}
\sim
\begin{bmatrix}
2 & 3 & 0 & -1 & -1 \\
0 & -1.5 & 3 & 1.5 & 7.5 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

(Note: There are many echelon forms for a given matrix; the specific steps that lead to the one given above are omitted here.) Only the first two rows of this echelon form have nonzero entries somewhere in the first four columns (the columns that correspond to the original matrix \(A\), with the lead nonzero entry (generally called a pivot) of the 1\(^{st}\) row appearing in the 1\(^{st}\) column, and the pivot in the 2\(^{nd}\) row appearing in the 2\(^{nd}\) column. With no pivots in the 3\(^{rd}\) or 4\(^{th}\) column, we might consider the corresponding variables \(x_3\) and \(x_4\) to be free, and assign them parameter values: \(x_3 = s\) and \(x_4 = t\). (Note: At this point we know that our parametrization will consist of rational functions in \(\mathbb{R}(s, t)\), and, correspondingly, that our linear variety is a surface.) Using the values for \(x_3\), \(x_4\), the echelon matrix says

\[
2x_1 + 3x_2 - t = -1 \\
-1.5x_2 + 3s + 1.5t = 7.5.
\]

The lower-triangular stair-step structure of echelon form suggests it is profitable to find the remaining unknowns \(x_1, x_2\) in reverse order—that is, the 2\(^{nd}\) equation gives us \(x_2 = 2s + t - 5\) and, after substituting this value in for \(x_2\) in the first equation, we have \(x_1 = -3s - t + 7\). So, we have arrived at the polynomial parametric representation of our linear variety

\[
x_1 = -3s - t + 7, \quad x_2 = 2s + t - 5, \quad x_3 = s, \quad x_4 = t.
\]
Parametrizations of affine (but not linear) varieties are not always there to be found and, even when they are, one does not use the same recipe in each instance to find them. Here are a couple of examples.

**Example 5.1**

Consider, again, the twisted cubic variety in $\mathbb{R}^3$ whose defining equations are

$$y - x^2 = 0 \quad \text{and} \quad z - x^3 = 0.$$  

Even though this is not a linear variety, we might co-opt some of the ideas developed when working with linear varieties above. In particular, there are $n = 3$ variables and $s = 2$ constraints, so we might expect our variety to be a curve (i.e., have 1 degree of freedom). In the linear case, we “doled out” that freedom to $n - s$ of the variables by making them parameters, and solved for the remaining non-free variables. Looking at the defining equations, it would be easy in the one to solve for $y$ in terms of $x$, and easy in the other to solve for $z$ in terms of $x$. Thus, we might assign $x = t$ (i.e., it serves as the parameter and holds all the freedom), and arrive at the parametrization

$$x = t, \quad y = t^2, \quad z = t^3.$$  

In SAGE we can plot (a part of) this affine variety quite easily with

```python
parametric_plot3d((x, x^2, x^3), (x, -5, 5))
```

**Example 5.2**

Consider the ellipse

$$\frac{x^2}{16} + y^2 = 1,$$

an affine variety in $\mathbb{R}^2$. One parametrization is

$$x = 4 \cos t, \quad y = \sin t.$$  

Try obtaining another parametrization following the example done in Section 1.3 that gives rise to the parametrization

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}$$  

for the unit circle.