The Locker Problem

Instructions. Work on the locker problem described below. As you work on the locker problem, have one member of your group keep track of the following information in a form that you can hand in at the next class period.[3]

- Collaborators: Who is in your group?
- Understanding the Problem: Describe any things you did to make sure *everyone* in the group understands the problem. Also list any misundersandings you may have had.
- Strategies: Make a list of each problem solving strategy you consider using to solve the problem. For each say why you decided to use or not use the strategy, and if you used it, record the work you did using that strategy. Include also strategies that someone suggested but that you did not pursue along with a reason why you did not pursue it (didn't know how to apply it to this problem, thought another strategy looked more promising, etc.).
- Conjectures: Make a list of things that you think might be true but for which you do not have a convincing argument. Include with this the *evidence* that makes you believe your conjecture.
- Findings: Make a list of things that you are able to show are true or false. Include a convincing argument for each finding.
- Looking Back: Whether you solve the problem "completely" or not, look back over you work periodically. Ask yourselves questions like: Have we overlooked anything? Do we have a goal? Are we making progress toward our goal? Is our solution really "complete"?
- Miscellaneous: Make note of anything else that you discover about the problem, yourself, mathematics, etc., which is not covered in the headings above.

1. The Locker Problem. One night the janitors at a local high school were bored, so they decided to play a little game. The noticed that there were exactly 100 lockers at the school, all of which were closed. The first janitor went down the halls and opened every locker. Let's call that round 1. Then the second janitor went down the halls and closed every second locker (lockers 2, 4, 6, dots, 98, 100). Let's call that round 2. In round 3, the first janitor went down the halls again and switched the door (opened it if had been closed, closed it if it had been open) of every third locker (lockers 3, 6, …, 96, 99). In round 4, the other janitor changed the door on every fourth locker (lockers 4, 8, 12, …, 96, 100). This process continued for 100 rounds until in the last round, one of the janitors changed only the 100th locker and both janitors went home.

Which lockers were open at the end of all this? Why?

[3] To spread the work around a bit, you might like to appoint one person to be "secretary" during class and a different person to take responsibility for writing things up to turn in next time.

Sieve of Eratosthenes

Eratosthenes is credited with being the first person to use a "prime number sieve" to identify prime numbers. His idea can be illustrated using a 100-board:

1. First cross out 1, since it is not prime.
2. Circle 2, and shade the upper left corner of all multiples of 2.
3. Circle 3 (which to this point has no markings in its box), and shade the upper right corner of all multiples of 3.
4. Circle 5 (which to this point has no markings in its box), and shade the lower right corner of all multiples of 5.
5. Circle the next number that to this point has no markings in its box, and shade the lower left corner of all multiples of this number.

Now answer the following questions about what you just did.

1. What is the smallest number with no markings in its box? If you marked all of the multiples of this number, would any new numbers receive their *first* mark? (Do not actually make the marks, just determine if any new numbers would receive their first mark.)

2. Describe the geometric pattern made by the multiples of 2 (marked in upper left corner) on the 100 board.

3. Describe the geometric pattern made by the multiples of 3 (marked in upper left corner) on the 100 Board.

4. What do the circled numbers represent? What do the unmarked numbers represent? Why?

5. Use your 100-board to determine which primes divide 84. (These numbers are called the prime factors of 84.) Does 84 have any factors that are not prime?

6. How can you identify all the multiples of 6 from the markings on your chart? (Do not add any additional markings.)

7. How can you identify all numbers that have both 6 and 7 as factors (divisors) from the markings on your chart? (Do not add any additional markings.)

8. Can you tell from the markings on your chart which numbers have 14 for a factor (divisor)? What about 9? What about 20? Explain. (Do not add any additional markings to your chart.)

9. Suppose you decided to use the sieve method to find all prime numbers less than 400 using a "400 Board" (with numbers 1 through 400). Multiples of which additional numbers would need to be marked in some way? Explain.

10. Do you think this is a good way to tell if large numbers are prime?

100 Board

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

A Factor Game

RULES. The game is played on a 30-board like the one below. Each player chooses a type of marker (pennies and nickels, colored chips, etc). Players take turns doing the following:

- Select an unmarked number from 1 – 30 that has at least one *unmarked* proper factor.
- Mark the selected number for yourself.
- Mark all unmarked factors of the selected number for your opponent. (There must always be at least one for your opponent or you are making an illegal move.)

This continues until there are no more legal moves to be made. The score is computed by adding up all the marked numbers for each player. The winner is the player with the largest sum.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

Analysis of the Factor Game

1. Play the game at least two times with a partner, recording the moves each player makes and keeping score as you go.

2. Begin an analysis of the game by considering the first two moves (one by each player).

(a) What number do you think the first player should choose? Why?

(b) How should the second player respond? Why?

3. Variations. Consider the following variations of the game:

(a) Work together to reach the end of the game in as few moves possible. Record your shortest game. (Record the numbers selected by each player on each turn).

(b) Work together to delay the end of the game for as many moves possible. Record your longest game. (Again, record the numbers selected by each player on each turn).

(c) How does the game change if instead of adding the numbers marked, you just count the number of markers to determine the score.

4. Formulate some good questions about this game. Also formulate a possible answer. (You do not have to have any idea whether your answer is correct or not, but it will help you formulate the question and think about what the answer would be like to write down an answer anyway.)

Some Problems Involving Number Theory

1. Mrs. Trubblemacher hosted a party for her son's Boy Scout troop. She was quite flustered having a house full of enthusiastic boys, so she never got an exact count of how many kids attended the party. However, she did know that she made sure that every boy got the same number of juice boxes, and all 3 dozen boxes she had were used. She also knew that they ate the 90 cookies she had on hand, and that she was very careful that every boy got the same number of cookies. (Mrs. Trubblemacher was well aware of the high priority scouts place on fairness!)

a) It seemed to Mrs. Trubblemacher that quite a few boys attended the party. What is the largest number of boys that could have attended the party?

b) What other numbers of boys are possible?

2. Sally's favorite meal combination at the school cafeteria consists of pizza, sliced pears, and carrot sticks. Pizza is served every eighth school day, sliced pears are served every sixth school day, and carrot sticks are served every third school day. She enjoyed her favorite meal Monday. How long will it be before Sally's favorite combination is served again?

3. If you have 36 little square tiles, how many different ways can they be arranged to make rectangles (that use all 36 tiles)?

4. The kids attending Risenshine Summer Camp can be split up into pairs for canoeing, teams of 4 for tennis, teams of 5 for basketball, and teams of 9 for softball. In each case, there are no campers left over. How many Risenshiners are there?

5. How many factors does 320 have?

6. a) What is the smallest number having exactly 8 factors?

b) What is the smallest number having exactly 7 factors?

c) Is there a general method for finding numbers with any specified number of factors?

7. I sometimes play basketball at noon on Tuesdays and Thursdays. One day we had 13 players split into a team of six and a team of seven; five players from each team would be playing at any one time, and the "extras" for each team would rotate into the game as substitutes every 4 minutes. I was having a heated discussion on the sideline with Bob about whose students were more responsible; Bob was one of the 2 subs for the 7-person team at the time, and I was the sub for the 6-person team. In the middle of our discussion, we were interrupted because it was my turn to rotate in, along with the other sub on Bob's team; Bob had to wait 4 more minutes before he got to play.

a) How many minutes passed before Bob and I were both "sitting out" again at the same time, so we could continue our discussion?

b) What if the 7-person team rotated their subs in every 2 minutes (and my team kept a 4-minute substitution plan)?

c) What if the 7-person team rotated two subs every 4 minutes; and the 6-person team, one sub every 4 minutes.

Common Divisors & Common Multiples

Fill out the following chart. Do you see any patterns? Can you explain them?

	prime factorization	all divisors (underline common divisors, circle gcd)	some multiples (underline common multiples, circle lcm)
9 12	$3 \times 3 = 3^2$ $2 \times 2 \times 3 = 2^2 \times 3$	$\underline{1}, \boxed{3}, 9$ $\underline{1}, 2, \boxed{3}, 4, 6, 12$	$9, 18, 27, \boxed{36}, 45, 54, 63, \underline{72}$ $12, 24, \boxed{36}, 48, 60, \underline{72}, \ldots$
15 21			
84 90			
70 105			
72 108			
56 225			
72 108 240			

Notes:

- GCD stands for 'greatest common divisor.' LCM stands for 'least common multiple.'
- Factors and diviors are the same thing. Some people refer to the gcd as gcf.

Cuisenaiare Rods as a Model for Fractions

Using the "whole-part-fraction" interpretation of fractions, fill in the following chart using Cuisenaire blocks. Note: there is one that is not possible with this model.

Whole	Part	Fraction
dark green		$\frac{1}{3}$
brown		$\frac{1}{4}$
orange		$\frac{2}{5}$
dark green		$\frac{5}{4}$
dark green		$1+\frac{2}{3}$
blue	light green	
orange + yellow	dark green	
light green	purple	
orange	dark green	
brown	yellow	
brown	orange	
	red	$\frac{1}{4}$
	dark green	$\frac{3}{5}$
	yellow	$\frac{1}{2}$
	dark green	$1+\frac{1}{2}$
	red	$\frac{2}{3}$
light green	yellow	
yellow	light green	
orange		$\frac{4}{5}$
	blue	$\frac{3}{4}$
	yellow	$\frac{5}{2}$
orange	purple	
	orange	$\frac{5}{2}$