# Chapter 2

# Cardinality

Most of the proofs (with one or two exceptions) in this chapter are fairly straightforward exercises or can be proven using one of two Cantor techniques: the zig-zag proof of countability or the diagonalization proof of uncountability. In Section 2.1 we will list the definitions and results. In chapter 2.2, each of Cantor's techniques will be illustrated. The remaining proofs are left as exercises.[1]

## 2.1 Basic Cardinality Results

We begin with a definition which says that two sets are the same size if we can "pair up" the elements of one set with the elements of the other.

**Definition 2.1.1 (Same-sized sets)** *Two sets $A$ and $B$* are the same size *(written $|A| = |B|$) if there is a function $f : A \to B$ such that*

- *$f$ is one-to-one (that is, for any $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1 \neq a_2)$), and*
- *$f$ is onto (that is, for any $b \in B$, there is an $a \in A$ such that $f(a) = b$).*

**Example 2.1.2** *Let $\mathbb{N}$ be the set of natural numbers $\{0, 1, 2, 3, \ldots\}$, and let $E$ be the non-negative even integers ($E = \{0, 2, 4, 6, 8, \ldots\}$). Then $|\mathbb{N}| = |E|$, since $f : n \mapsto 2n$ is one-to-one and onto.*

If when we pair up the elements of $A$ with elements of $B$ we use up all of $A$ but perhaps have some elements of $B$ "left over", then $A$ cannot be larger than $B$:

[1] Note: We are taking the approach here of "naive set theory", that is assume that sets behave as you expect them to. Later we will revisit set theory in a more formal way. Everything we do here will be valid even in that context, but a couple of these proofs require the Axiom of Choice. For now, it will be used without comment. Perhaps later when we revisit set theory we will comment on the role this axiom plays.

**Definition 2.1.3 (No bigger than)** *A set A is* no bigger than *the set B (written $|A| \leq |B|$), if there is a function $f : A \to B$, such that*

- *$f$ is one-to-one.*

The notation chosen above suggests that "same size as" and "no bigger than" behave in nice ways. The next two theorems demonstrate that this is the case:

**Theorem 2.1.4 (Properties of =)** *"Same size as" is an equivalence relation.*[2]

1. *"Same size as" is reflexive: $|A| = |A|$.*
2. *"Same size as" is symmetric: If $|A| = |B|$, then $|B| = |A|$.*
3. *"Same size as" is transitive: If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.*

**Theorem 2.1.5 (Properties of $\leq$)**

1. *"No bigger than" is reflexive: $|A| \leq |A|$.*
2. *"No bigger than" is transitive: If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.*
3. *If $A \subseteq B$, then $|A| \leq |B|$.*
4. *If there is a function $f : A \to B$ that is onto, then $|B| \leq |A|$.*
5. *If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Thus the use of = and $\leq$ is justified in our notation. This also suggests some extensions to the notation:

- $|B| \geq |A|$ means $|A| \leq |B|$,
- $|A| \neq |B|$ means it is not the case that $|A| = |B|$.
- $|A| < |B|$ means $|A| \leq |B|$ but $|A| \neq |B|$.

Do note, however, that we have defined $|A| = |B|$ and $|A| \leq |B|$ as *predicates*. We have not defined $|A|$ as a function. Officially $|A|$ is nonsensical. There is a way to make it make sense, but it requires further study of set theory (and the Axiom of Choice) to do so.

**Definition 2.1.6 (Finite, infintite)** *A set A is* infinite *if there is a subset $B \subsetneq A$ such that $|A| = |B|$. Otherwise, A is* finite.

[2] An equivalence relation is a relation that is reflexive, symmetric, and transitive.

**Definition 2.1.7 (Countable)** *If $|A| = |\mathbb{N}|$, then we say that $A$ is* countably infinite. *A* countable *set is any set that is either finite or countably infinite. In other words, a countable set is the same size as some subset of $\mathbb{N}$.*

Most of the important infinite sets we will encounter in this class will be countably infinite. The following theorem establishes some nice properties of countable sets.

**Theorem 2.1.8 (Properties of countable sets)**

1. *A finite union of countable sets is countable: Suppose that $A$ and $B$ are countable. Then $A \cup B$ is also countable.*

2. *A countable union of countable sets is countable: Suppose that for each natural number $n$, $A_n$ is countable. Then $A = \cup_{n=0}^{\infty} A_n$ is also countable.*

3. *The cross product of countable sets is countable: Let $A$ and $B$ be countable, then $A \times B$ is countable.*

4. *The set of all finite sequences from a countable set is countable: Let $A^{=n}$ denote the set of all sequences of $n$ items from $A$. (For example, $(1, 4, 3, 0) \in \mathbb{N}^{=4}$.) Then $\cup_{n=0}^{\infty} A^{=n}$ is countable.*

5. *The following sets are all countable: $\mathbb{N}$, $\mathbb{Z}$ (the integers), $\mathbb{Q}$ (the rationals), the set of even integers.*

6. *The following sets are countable: the set of all FOL sentences (in a fixed, countable language), the set of all proofs (again in a fixed, countable language).*

7. *The following sets are uncountable: $\mathbb{R}$ (the real numbers), $[0, 1]$ (the reals in the interval from $0$ to $1$).*

8. *Every infinite set has a countably infinite subset.*

Finally, Cantor's diagonalization argument can be used to establish the following general fact:

**Theorem 2.1.9 (Power set)** *Let $\mathcal{P}(A)$ denote the power set of $A$ (i.e., the set of all subsets of $A$). Then $|A| < |\mathcal{P}(A)|$.*

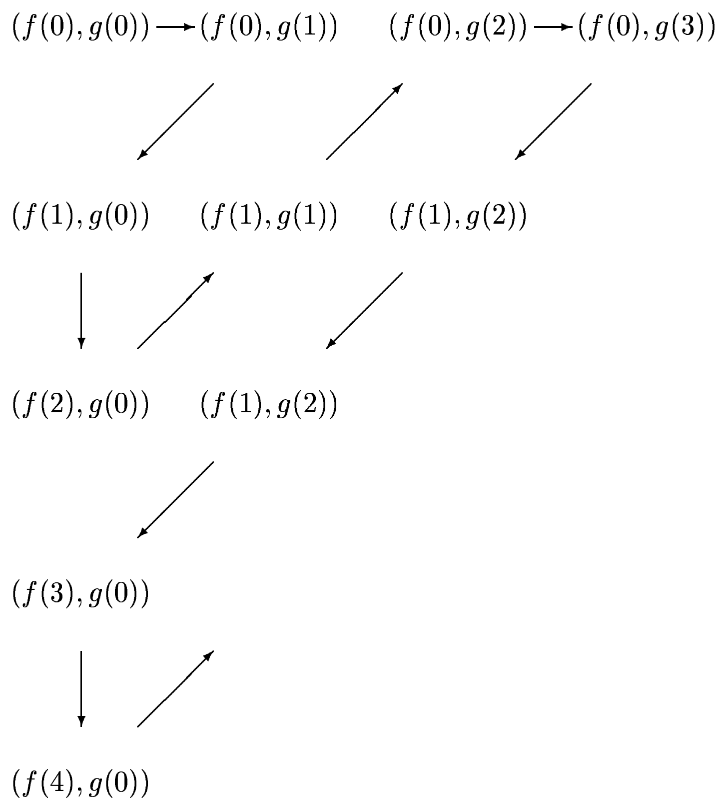
## 2.2 Cantor's Two Ideas

### 2.2.1 Zig-Zag

Several of the countability results of the previous chapter can be proven using a zig-zag argument that goes back to Cantor. We will use this method here to prove part 3 of Theorem 2.1.8.

*Proof (of Theorem 2.1.8, part 3).* Let $A$ and $B$ be countably infinite. We must show that $A \times B$ is countably infinite. (Strictly speaking, we need to deal with the cases where one or both of the sets are finite, too, but we will only do the case where both are infinite here.)

Since $A$ and $B$ are countable, there are one-to-one, onto functions $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. So $A \times B = \{(f(i), g(j)) | i, j \in \mathbb{N}\}$. The key idea of Cantor is to arrange the elements of $A \times B$ in a rectangular grid filling one quadrant of the plane:




As the picture indicates, we can enumerate $A \times B$ beginning in the upper left-hand corner and following the arrows. In fact, by slightly modifying the zig-zag argument (travel each diagonal from top to bottom instead of zig-zagging), it is not too hard to give an exact formula for a one-to-one, onto function mapping $A \times B$ to $\mathbb{N}$ (or vice versa). $\square$

It is worthwhile to give another proof of this.

*Proof (of Theorem 2.1.8, part 3).* This time we will make use of part 5 of Theorem 2.1.5. For this we need to exhibit one-to-one functions $\alpha : A \times B \to \mathbb{N}$ and $\beta : \mathbb{N} \to A \times B$. The following two functions can easily be shown to be one-to-one (for the first we use the fact that prime factorizations are unique):

$$\begin{aligned} \alpha &: (f(i), g(j)) \mapsto 2^i 3^j \\ \beta &: n \mapsto (f(n), g(0)) \end{aligned}$$

□

### 2.2.2 Diagonalization

Cantor's diagonalization idea is even cleverer than the previous idea. We will use it here to prove Theorem 2.1.9.

*Proof (of Theorem 2.1.9).* Let $A$ be any set, we need to show that $|A| < |\mathcal{P}(A)|$. First notice that clearly $|A| \leq |\mathcal{P}(A)|$, since

$$x \mapsto \{x\}$$

is one-to-one.

The heart of the matter is to show that there is no function $f : A \to \mathcal{P}(A)$ that is onto. We will do this using the method of "defeating an arbitrary example". For any function $f : A \to \mathcal{P}(A)$, we will describe a method to show that it is not onto. That is, for any such function $f$, we must find some subset $S_f$ of $A$ that gets "missed" by the function $f$. One such set is

$$S_f = \{a \in A | a \notin f(a)\} .$$

Since for every $a$, $a \in S_f \iff a \notin f(a)$, we see that $S_f \neq f(a)$; that is, $S_f$ is missed by the function $f$. Since we have not assumed anything special about $f$, this shows that no function $f$ can be onto. Therefore $|A| \neq |\mathcal{P}(A)|$. □

## 2.3 Exercises

1. Prove Theorem 2.1.4.
2. Prove Theorem 2.1.5.
3. Prove Theorem 2.1.8.
4. Show that the set of irrational numbers is uncountable.
5. Which of the following are countable, which are uncountable?
   (a) The set of all functions from $\mathbb{N}$ to $\mathbb{N}$.
   (b) The set of all one-to-one functions from $\mathbb{N}$ to $\mathbb{N}$.
   (c) The set of all functions from $\mathbb{N}$ to $\mathbb{N}$ that are eventually 0.
   (d) The set of all functions from $\mathbb{N}$ to $\mathbb{N}$ that are eventually constant.