Math 362
Spring, 2002

Name ______________________________

Test 2

Instructions. The items marked with ♢ must be done in class. Do as much of the remainder of the test as you have time for in class and then redo those items as the take home portion of your test.

1. ♢a) Define uniform convergence of a sequence of functions $\{f_n\}$ on a metric space X.

♢b) Prove that if $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous.

c) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and that there is a number M such that for all $x \in \mathbb{R}$, $|f'(x)| < M$. Show that f is uniformly continuous on $\mathbb{R}$.

2. For each of the following, give an example of a sequence of functions $\{f_n\}$ with the listed properties. (No proofs are needed here, just examples).

♢a) each f_n is continuous, and $\sum f_n$ converges, but $\sum f_n$ is not continuous

♢b) each f_n is differentiable, and $\{f_n\} \to f$ uniformly, but $\{f_n'\} \not\to f'$.

♢c) each f_n is continuous, but

$$\lim_{n\to\infty} \left(\int_a^b f_n(x)\, dx \right) \neq \int_a^b \left(\lim_{n\to\infty} f_n(x) \right) dx$$

3. ♢a) Carefully state the Weierstrass Theorem.

♢b) Outline the proof of the theorem (no more than 1/2 page).

4. Let $f_n : \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$.

♢a) Let $a \in \mathbb{R}$. Under what conditions do we know that $\lim\limits_{x\to a} \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \lim\limits_{x\to a} f_n(x)$?

b) Under those same conditions (appropriately adjusted if necessary to handle ∞), is it also the case that $\lim\limits_{x\to\infty} \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \lim\limits_{x\to\infty} f_n(x)$? Prove or give a counterexample.

5. ♢a) Prove the following theorem:

Theorem. Suppose that $f_n : [a, b] \to \mathbb{R}$ for all $n \in \mathbb{N}$ and that $\{f_n\} \to f$ uniformly on $[a, b]$. Then f is integrable and

$$\int_a^b \lim_{n\to\infty} f_n(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx$$

b) Recall that

$$\int_a^\infty g(x)\, dx = \lim_{b\to\infty} \int_a^b g(x)\, dx$$

Replace $[a, b]$ with $[a, \infty)$ in the theorem above. Is the theorem still true? Prove it or give a counterexample.

6. Let $f_n : [0, 1] \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{1}{2^n}(nx - \lfloor nx \rfloor)$$

where $\lfloor x \rfloor$ is the greatest integer not larger than x. (So $\lfloor 4 \rfloor = 4$; $\lfloor 4.2 \rfloor = 4$; $\lfloor 3.9 \rfloor = 3$.)

a) Sketch graphs of the first few functions in the sequence.

b) Determine $\lim_{n\to\infty} f_n(x)$. Is the convergence uniform?

c) Let $E = \{x \mid \sum f_n(x) \text{ converges pointwise at } x\}$. What is E?

d) Does $\sum f_n(x)$ converge uniformly on E?