Basic Properties of Measureable Sets

Math 362 – Spring 2002

R. Pruim

rpruim@calvin.edu

Calvin College

Usage Notes

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Thm 11.10

Thm. $\mathcal{M}(\mu)$ is a σ-ring and μ^* is countably additive on $\mathcal{M}(\mu)$.

Proof. The proof consists of establishing 7 properties:

1. $\mathcal{M}_{\mathcal{F}}(\mu)$ is a ring.
2. If $A_n \to A$ and each $A_n \in \mathcal{E}$, then $\mu^*(A_n) \to \mu^*(A)$.
3. μ^* is additive on $\mathcal{M}_{\mathcal{F}}(\mu)$.
4. $\mu^*(\cup A_n) = \sum \mu^*(A_n)$ if A_n are disjoint, elementary sets.
5. If $A \in \mathcal{M}(\mu)$ and $\mu^*(A) < \infty$, then $A \in \mathcal{M}_{\mathcal{F}}(\mu)$.
6. μ is countably additive on $\mathcal{M}(\mu)$.
7. $\mathcal{M}(\mu)$ is closed under countable unions and difference (so

1. $\mathcal{M}_{\mathcal{F}}(\mu)$ is a ring

Let $A, B \in \mathcal{M}_{\mathcal{F}}(\mu)$. Then there are elementary sets $\{A_n\}$ and $\{B_n\}$ such that $A_n \to A$ and $B_n \to B$.

We can show that that $A \cup B \in \mathcal{M}_{\mathcal{F}}(\mu)$

by showing that $A_n \cup B_n \to A \cup B$

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Note that

$$\begin{aligned} d(A_n \cup B_n, A \cup B) &= \mu^*((A_n \cup B_n) \triangle (A \cup B)) \quad (3) \\ &\leq \mu^*((A_n \triangle A) \cup (B_n \triangle B)) \quad (3) \\ &\to 0 \text{ as } n \to \infty \quad (3) \end{aligned}$$

reason for (2): $(A_n \cup B_n) \triangle (A \cup B) \subseteq (A_n \triangle A) \cup (B_n \triangle B)$ (check using a membership table.)
The proof for differences is similar.

2. $\mu^*(A_n) \to \mu^*(A)$

(where $A_n \to A$ and each $A_n \in \mathcal{E}$)

- $\mu^*(A) \leq \mu^*(A_n) + \mu^*(A \triangle A_n)$ (by sub-additivity of μ^*), so
- $\mu^*(A) - \mu^*(A_n) \leq \mu^*(A \triangle A_n) \to 0$

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Similarly (reversing roles of A and A_n) we see that

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Combining these we get

- $|\mu^*(A_n) - \mu^*(A)| \to 0$ as $n \to \infty$.

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Note: This implies that $\mu^*(A) < \infty$ for all $A \in \mathcal{M}_{\mathcal{F}}(\mu)$.

3. μ^* is additive on $\mathcal{M}_{\mathcal{F}}(\mu)$

Let $A_n \to A$ and $B_n \to B$ be convergent sequences of elementary sets.

We know: $\mu(A_n) + \mu(B_n) = \mu(A_n \cup B_n) + \mu(A_n \cap B_n)$

Letting $n \to \infty$ we get:

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- $\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B)$.

  Recall that we just showed that $\lim \mu^*(A_n) = \mu^*(A)$ if $A_n \to A$ and each $A_n \in \mathcal{E}$. So for any $\varepsilon > 0$, if we choose n large enough,

  $|\mu^*(A_n) - \mu^*(A)| < \varepsilon, \quad |\mu^*(B_n) - \mu^*(B)| < \varepsilon$,

  $|\mu^*(A_n \cup B_n) - \mu^*(A \cup B)| < \varepsilon$,

  $|\mu^*(A_n \cap B_n) - \mu^*(A \cap B)| < \varepsilon$,

  So $|\mu^*(A) + \mu^*(B) - \mu^*(A \cup B) - \mu^*(A \cap B)| < 4\varepsilon$.

3. μ^* is additive on $\mathcal{M}_{\mathcal{F}}(\mu)$

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We know: $\mu(A_n) + \mu(B_n) = \mu(A_n \cup B_n) + \mu(A_n \cap B_n)$

Letting $n \to \infty$ we get:

- $\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B)$.

If $A \cap B = \emptyset$, then $\mu^*(A \cap B) = 0$, so $\mu(A) + \mu(B) = \mu(A \cup B)$.

4. $\mu^*(\cup A_n) = \sum \mu^*(A_n)$

(if A_n are disjoint, elementary sets)

$\boxed{\mu^*(\cup A_n) \leq \sum \mu^*(A_n)}$: follows from sub-additivity

$\boxed{\mu^*(\cup A_n) \geq \sum \mu^*(A_n)}$: Compare $\mu^*(\cup A_n)$ with $\sum_{k=1}^{n} \mu^*(A_k)$

- $A = \cup A_n \supseteq A_1 \cup A_2 \cup \cdots A_n$, so
- $\mu^*(A) \geq \mu^*(A_1) + \mu^*(A_2) + \cdots \mu^*(A_n)$, so
- $\mu^*(A) \geq \lim_{n\to\infty} \mu^*(A_1) + \mu^*(A_2) + \cdots \mu^*(A_n) = \sum \mu^*(A_n)$.

• $A \in \mathcal{M}(\mu)$ & $\mu^*(A) < \infty \Longrightarrow A \in \mathcal{M}_{\mathcal{F}}(\mu$

Let $A = \cup A_n$ where $A_n \in \mathcal{M}(\mu)$. WLOG the $A_n's$ are disjoint.

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Let $A = \cup A_n$ where $A_n \in \mathcal{M}(\mu)$. WLOG the $A_n's$ are disjoint.

- If $A = \cup \hat{A}_n$, then define $A_n = \hat{A}_n - (A_1 \cup A_2 \cup \cdots \cup A_n)$ to get a disjoint union.

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Let $B_n = A_1 \cup A_2 \cup \cdots \cup A_n$, then $B_n \to A$

- $B_n \in \mathcal{M}_{\mathcal{F}}(\mu)$ since $\mathcal{M}_{\mathcal{F}}(\mu)$ is a ring of sets.
- $d(A, B_n) = \mu^*(\cup_{i=n+1}^{\infty} A_i) = \sum_{i=n+1}^{\infty} \mu^*(A_i)) \to 0$

• $A \in \mathcal{M}(\mu)$ **&** $\mu^*(A) < \infty \Longrightarrow A \in \mathcal{M}_{\mathcal{F}}(\mu$

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Let $B_n = A_1 \cup A_2 \cup \cdots \cup A_n$, then $B_n \to A$, so ...

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Let $B_n = A_1 \cup A_2 \cup \cdots \cup A_n$, then $B_n \to A$, so . . .

. . . $A \in \mathcal{M}_\mathcal{F}(\mu)$.

- We would be done if each $B_n \in \mathcal{E}$, but we only know $B_n \in \mathcal{M}_\mathcal{F}(\mu)$.
- So for each n there is $B_n^k \to B$ (as $k \to \infty$) with $B_n^k \in \mathcal{E}$
- Exercise: use this to show that $A \in \mathcal{M}_\mathcal{F}(\mu)$.

. $A \in \mathcal{M}(\mu)$ & $\mu^*(A) < \infty \Longrightarrow A \in \mathcal{M}_{\mathcal{F}}(\mu$

Let $A = \cup A_n$ where $A_n \in \mathcal{M}(\mu)$. WLOG the $A_n's$ are disjoint.

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- Exercise: use this to show that $A \in \mathcal{M}_{\mathcal{F}}(\mu)$.

6. μ is countably additive on $\mathcal{M}(\mu)$

Let $A = \cup A_n$ with $A_n \in \mathcal{M}(\mu)$ and $\{A_n\}$ disjoint.

Consider 2 cases:

1. $\mu(A_n) = \infty$ for some A_n
   - then $\mu(A) = \infty = \sum \mu(A_n)$
2. $\mu(A_n) < \infty$ for all A_n
   - then each $A_n \in \mathcal{M}_{\mathcal{F}}(\mu)$, so $\mu(A) = \sum \mu(A_n)$ by (4)

7. $\mathcal{M}(\mu)$ is σ-ring

We need to show that $\mathcal{M}(\mu)$ is:

- closed under countable unions
- closed under difference

7. $\mathcal{M}(\mu)$ is σ-ring

We need to show that $\mathcal{M}(\mu)$ is:

- closed under countable unions
  - Let $\{A_n\}$ be a countable collection of sets from $\mathcal{M}(\mu)$
  - So $A_n = \cup_{k>1} A_{n,k}$ where each $A_{n,k} \in \mathcal{M}_{\mathcal{F}}(\mu)$.
  - But then $A = \cup_{n>1} \cup_{k>1} A_{n,k}$ is a countable union of sets in $\mathcal{M}_{\mathcal{F}}(\mu)$, so $A \in \mathcal{M}(\mu)$
- closed under difference

7. $\mathcal{M}(\mu)$ is σ-ring

We need to show that $\mathcal{M}(\mu)$ is:

- closed under countable unions
- closed under difference
    - Let $A = \bigcup A_n$, $B = \cup B_n$ with $A_n, B_n \in \mathcal{M}(\mu)$.
    - $A - B = \cup(A_n - B)$, so it suffices to show $A_n - B \in \mathcal{M}_{\mathcal{F}}(\mu)$.
    - $A_n - B \subseteq A_n \in \mathcal{M}_{\mathcal{F}}(\mu)$, so it suffices to show that each $A_n - B \in \mathcal{M}(\mu)$.
    - $A_n - B = A_n - (A_n \cap B)$, so it suffices to show that $A_n \cap B \in \mathcal{M}(\mu)$.
    - $A_n \cap B = \cup(A_n \cap B_n) \in \mathcal{M}(\mu)$, so we're done.

Appendectory Notes Follow

Membership Table

A_n	B_n	$A_n \cup B_n$	A	B	$A \cup B$	$(A_n \cup B_n) \triangle (A \cup B)$	$(A_n \triangle A) \cup (B_n \triangle B)$
1	0	1	0	0	0	1	1
0	1	1	0	0	0	1	1
1	0	1	0	0	0	1	1
0	0	0	1	0	0	1	1
0	0	0	0	1	0	1	1
0	0	0	1	1	0	1	1

Note that in all missing rows of the table, there would be a 0 in the column labeled $(A_n \cup B_n) \triangle (A \cup B)$.

Return