Math 362 – Spring 2002
Homework Solutions

Problem 7.2b

Let $\{f_n\}$, and $\{g_n\}$ be sequences of bounded functions that converge uniformly to f and g on some set E. Thus for each n there is a real number M_n such that for all $x \in E$, $|f_n(x)| < M_n$, and $|g_n(x)| < M_n$. Since $\{f_n\}$, and $\{g_n\}$ converge uniformly, there is an $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$ and all $x \in E$,

- $|f_n(x) - f_m(x)| < 1$
- $|f_n(x) - f(x)| < 1$
- $|g_n(x) - g_m(x)| < 1$
- $|g_n(x) - g(x)| < 1$

From this (by the triangle inequality), we see that there is an M (namely, $M_{n_0} + 1$), such that for all $x \in E$ and all $n > n_0$:

- $|f_n(x)| < M$ (because $|f_n(x)| \leq |f_{n_0}(x)| + |f_{n_0}(x) - f_m(x)| < M_{n_0} + 1 = M$)
- $|f(x)| < M$
- $|g_(x)| < M$
- $|g(x)| < M$

Since there are only finitely many $n < n_0$, we can extend this to get a bound that works for all x and all n, namely $\max(\{M_n \mid n \leq n_0\} \cup \{M\})$, but this is not necessary.

Now let $\varepsilon > 0$ be given. If n is large enough, then for all $x \in E$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2M}$ and $|g_n(x) - g(x)| < \frac{\varepsilon}{2M}$, so

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\ &= |f_n(x)(g_n(x) - g(x)) + (f_n(x) - f(x))g(x)| \\ &\leq |f_n(x)| \cdot |g_n(x) - g(x)| + |f_n(x) - f(x)| \cdot |g(x)| \\ &< M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} \cdot M \\ &= \varepsilon \end{aligned}$$

Thus $\{f_n \cdot g_n\}$ converges uniformly to $f \cdot g$.

Problem 7.3

If we don't require boundedness in the previous problem, there are many examples where $\{f_n \cdot g_n\}$ does not converge uniformly. Perhaps the simplest is $f_n(x) = 1/n$ and $g_n(x) = x$. In this case $|f_n(x)g_n(x) - f(x)g(x)| = |x/n - 0|$ which approaches zero for every x as n increases, but not uniformly so, since for any n, $|f_n(n)g_n(n) - f(n)g(n)| = |n/n - 0| = 1$. Note that this example shows that it in the previous exercise it is important that both sequences to be bounded, since the result has been shown to fail with one bounded and the other not.

Problem 7.5

$f_n(x) \geq 0$ for all x, so convergence and absolute convergence are the same. Also notice that for all n, there is an x (namely $x_n = \frac{1}{n+\frac{1}{2}}$) such that $f_n(x) = 1$. Also, for any x, $f_n(x) = 0$ for all but at most one n.

So $f_n(x) \to 0$ as $n \to \infty$, and the partial sums of $\sum f_n(x)$ sum are particularly easy, since at most one term is non-zero. Clearly, this means $\sum f_n(x)$ converges for each x. But the convergence is not uniform, since for any n, $\sum_{k=1}^{n} f_n(x_{n+1}) = 0$ (but $\sum_{k=1}^{\infty} f_k(x_{n+1}) = 1$. In other words, $||f_n - f|| = 1$ for all n.

Problem 7.6

Let $a_n(x) = (-1)^n \frac{x^2+n}{n^2}$. $\sum_n a_n$ is an alternating series, and converges by the alternating series test since $\{|a_n|\} \searrow 0$.

For fixed x, let $S(x)$ be the sum of the series and let $s_n(x)$ be the nth partial sum. Then $|S(x) - s_n(x)| \leq |a_{n+1}(x)| \leq \frac{b^2+n}{n^2} \to 0$ as $n \to \infty$, provided $|x| \leq b$. So the convergence is uniform on any bounded interval.

On the other hand, $|a_n| > \frac{1}{n}$, and $\sum_n \frac{1}{n}$ diverges (p-series), so $\sum_n |a_n|$ diverges by the comparison test.

Problem 7.7

Let $f_n(x) = \frac{x}{1+nx^2}$. Then $f_n'(x) = \frac{-2nx^2}{(1+nx^2)^2} + \frac{1}{1+nx^2} = \frac{1-nx^2}{(1+nx^2)^2}$. As $n \to \infty$, $f_n(x) \to 0$ for all x, $f_n'(x) \to 0$ for all $x \neq 0$, and $f_n'(0) \to 1$. So $f_n'(0) \not\to f'(0) = 0$.

It remains to show that $\{f_n\}$ converges uniformly to 0. Notice that $f_n'(x) = 0$ only at $x = \pm\frac{1}{\sqrt{n}}$, and that $|f_n(\pm\frac{1}{\sqrt{n}})| = \frac{1}{2\sqrt{n}}$ is the maximum value of $|f_n(x)|$, so for all x, $|f_n(x) - 0| \leq \frac{1}{2\sqrt{n}}$. Thus for any $\varepsilon > 0$, if $n > \frac{1}{\varepsilon^2}$, then $|f_n(x) - 0| < \varepsilon$ for all x. So the convergence is uniform.

Problem 7.9

Let $\varepsilon > 0$ be given. Since f is continuous, there is a $\delta > 0$ such that $|x - t| < \delta$ implies

- $|f(x) - f(t)| < \varepsilon/2$.

Since $\{f_n\} \to f$ uniformly and $\{x_n\} \to x$, there is an $n_0 \in \mathbb{N}$ such that $n > n_0$ implies

- $|f_n(x) - f(x)| < \varepsilon/2$ for all x, and
- $|x_n - x| < \delta$

Putting this all together we see that if $n > n_0$ then

$$\begin{aligned} |f_n(x_n) - f(x)| &\leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

So $\lim_{n\to\infty} f_n(x_n) = f(x)$.

The converse is false. One counterexample is $f_n(x) = x/n$ on $E = (0, \infty)$. Problem 7.5 provides another example.

Cantor Set

Recall the the Cantor set $C = \cap E_n$, where E_n is a collection of 2^n disjoint intervals of width 3^{-n}. Let $\{x_1, x_2, \ldots\}$ be a sequence of elements of C. Let $I_0 = [0, 1]$. For each natural number $n > 0$, let I_n be one of the 2^n disjoint intervals that comprise E_n chosen so that

- $I_n \subset I_{n-1}$
- $x_n \notin I_n$

Such a choice can be made since $I_{n-1} \cap E_n$ consists of two disjoint intervals, and x_n cannot be in both of them.

Since each I_n is a nested sequence of non-empty, compact sets, $\cap I_n$ is not empty. Let $z \in \cap_{n=1}^{\infty} I_n$. Then $z \in C$ but $z \notin \text{range}(x)$. Therefore C is uncountable since we have shown that any sequence of points from the Cantor set fails to be onto.

The outer measure of C must be 0, since we can cover C with open intervals of arbitrarily small total measure as follows: Let $\varepsilon > 0$ be given. Choose n so $\frac{2^n}{3^n} < \frac{\varepsilon}{2}$, and consider open consisting of all intervals of the form $(a - \frac{\varepsilon}{2^{n+2}}, b + \frac{\varepsilon}{2^{n+2}})$, where $[a, b]$ is one of the 2^n intervals comprising E_n. Then the total measure of this cover is $\frac{2^n}{3^n} + 2 \cdot 2^n \cdot (\frac{\varepsilon}{2^{n+2}}) < \varepsilon$.

Problem 11.1

Suppose $f \geq 0$ and $\int f \, d\mu = 0$. Let $E = \{x | f(x) > 0\}$, let $E_n = \{x | f(x) > \frac{1}{n}\}$, and let s_n be the simple function defined by $s_n(x) = \frac{1}{n} \cdot K_{E_n}(x)$. Then $s_n(x) < f(x)$, for all x, so $0 \leq \int s_n \, d\mu \leq \int f \, d\mu = 0$. Since $0 = \int s_n \, d\mu = \frac{1}{n}\mu(E_n)$, we know that $\mu(E_n) = 0$ for all n. But $\mu(E) = \mu(\cup E_n) = \lim_{n \to infty} \mu(E_n) = 0$ (by Thm. 11.3).

Problem 11.2

Let $A_1 = \{x \in E | f(x) \geq 0\}$ and let $A_2 = \{x \in E | f(x) < 0\}$. By hypothesis, $\int_{A_1} f \, d\mu = 0$ and $\int_{A_2} f \, d\mu = 0$, so also $\int_{A_2} -f \, d\mu = 0$. Since $f \geq 0$ on A_1 and $-f \geq 0$ on A_2, $f = 0$ a.e. on each of A_1 and A_2 (by problem 11.1 , hence $f = 0$ a.e. on $E = A_1 \cup A_2$. (For this last claim we are using the facts that $\{x \in E | f(x) \neq 0\} = \{x \in A_1 | f(x) \neq 0\} \cup \{x \in A_2 | f(x) \neq 0\}$ and that the measure zero sets area ring (in fact a σ-ring).

Problem 11.3

Let $E = \{x \mid \{f_n(x)\} \text{ converges}\}$. Then since convergencent sequences in $\mathbb{R}$ are Cauchy, we have

$$\begin{aligned} E &= \{x \mid \forall M \exists N \ (i > j > N \Longrightarrow |f_i(x) - f_j(x)| < \frac{1}{M})\} && (1) \\ &= \bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \{x \mid i > j > N \Longrightarrow |f_i(x) - f_j(x)| < \frac{1}{M}\} && (2) \\ &= \bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{j=N}^{\infty} \bigcap_{i=j}^{\infty} \{x \ : \ |f_i(x) - f_j(x)| < \frac{1}{M}\} && (3) \end{aligned}$$

Now since for fixed i and j, $|f_i - f_j|$ is measurable, we see that the sets in (3) are measurable.