Test 3 Solutions

Please label your work clearly and leave 1 inch margins on all four edges of your paper. Be sure to follow directions.

1. a) Give a careful definition of $f'(x_0)$ (the derivative of $f$ at the point $x_0$).

b) Let $f(x) = x^2 + x$. Use the definition of derivative to determine $f'(1)$.

Let $T(x) = \frac{f(x) - f(1)}{x - 1}$ (for $x \neq 1$). Then $T(x) = \frac{x^2 + x - 2}{x - 1} = \frac{(x - 1)(x + 2)}{x - 1} = x + 2$.

So $f'(1) = \lim_{x \to 1} T(x) = 3$.

c) Let $g(x) = |x|$. Show that $g$ is not differentiable at $x = 0$.

Let $T(x) = \frac{|x| - 0}{x - 0}$ (for $x \neq 0$). If $x > 0$, then $T(x) = 1$, and if $x < 0$, then $T(x) = -1$.

So $\{T(1/n)\}_{n=1}^{\infty}$ converges to 1, and $\{T(-1/n)\}_{n=1}^{\infty}$ converges to -1. Thus $\lim_{x \to 0} T(x)$ does not exist, so $g$ is not differentiable at 0.

2. Show that if $f$ is differentiable at $c$, then $f$ is continuous at $c$. [Hint: Use Limit Blah Laws carefully.]

Suppose $f$ is differentiable at $c$. Then $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists. Since $\lim_{x \to c} (x - c) = 0$, by LBL $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot (x - c) = \lim_{x \to c} f(x) - f(c) = 0$. Applying LBL again, we see that $\lim_{x \to c} f(x) = \lim_{x \to c} (f(x) - f(c)) + f(c) = f(c)$, thus $f$ is continuous at $c$.

3. a) Carefully state the Mean Value Theorem.

b) Use the Mean Value Theorem to prove the following:

If $f : [a, b] \to \mathbb{R}$ is differentiable and $f'(x) \neq 0$ for all $x \in [a, b]$, then $f$ is one-to-one. [Hint: Contra-positive.]

Suppose $f$ is not one-to-one. Then there are $x$ and $y$ such that $f(x) = f(y)$. Then by MVT, there must be a $c \in (x, y)$ with $f'(c) = \frac{f(x) - f(y)}{x - y} = 0$.

4. Prove one of the following facts about Riemann integration. The first is worth more points.

a) If $f : [a, b] \to \mathbb{R}$ is a continuous function, then $f$ is Riemann integrable on $[a, b]$. [Hint: Use upper and lower sums.]

b) If $f : [a, b] \to \mathbb{R}$ is an increasing function, then $f$ is Riemann integrable on $[a, b]$. [Hint: Use upper and lower sums.]
The function $f$ is uniformly continuous, so for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{b - a}$. Let $P$ be any partition with mesh less than $\delta$. Since $f$ is continuous, by the EVT there are points $t_i$ and $s_i$ in the interval $[x_{i-1}, x_i]$ such that $f(t_i) = M_i$ and $f(s_i) = m_i$. So

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} (f(t_i) - f(s_i))\Delta x_i < \frac{\varepsilon}{b - a} \sum_{i=1}^{n} \Delta x_i = \varepsilon$$

which shows that $f$ is integrable.

5. True or False.

a) Indicate (by placing a T or F in the left margin next to each item) which statements are true and which are false about a function $f : [0, 1] \to \mathbb{R}$.

i. If $f'$ is bounded, then $f$ is bounded.

True. Let $K$ be such that $|f'(x)| < K$ for all $x \in [a, b]$. By MVT, there is a number $c$ such that $|f(x) - f(0)| = |f'(c)|(x - 0) < K \cdot 1 = K$, so $f$ is bounded by $f(0) - K$ and $f(0) = K$.

Alternatively, note that differentiable implies continuous and apply the Extreme Value Theorem.

Note that the first approach works even if $f$ is not differentiable at the endpoints of $[0, 1]$.

ii. If $f$ is bounded, then $f'$ is bounded.

False. Let $f(x) = \begin{cases} \sin(1/x) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

iii. If $f$ is Riemann integrable on $[0, 1]$, then $f^2$ is Riemann integrable on $[0, 1]$.  

True. Products of integrable functions are integrable.

iv. If $f^2$ is Riemann integrable on $[0, 1]$, then $f$ is Riemann integrable on $[0, 1]$.

False. Let $f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$

v. If for every partition of $[0, 1]$, $U(P, f) > 0$, then $\int_{0}^{1} f dx > 0$.

False. Let $f(x) = \begin{cases} 0 & \text{when } x > 0 \\ 1 & \text{when } x = 0 \end{cases}$

b) For each item that is false, give a counterexample. (You do not need to give a proof that the counterexample is a counterexample.)

c) For each item that is true provide a proof. You may refer to any theorems we have proven in class, even if this makes the problem very easy.

6. (Extra Credit) Find an example of a function $f : [0, 1] \to \mathbb{R}$, such that $f(x) > 0$ for all $x \in [0, 1]$, and $\int_{0}^{1} f dx = 0$. Is your function integrable?

Let $f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ is a rational in lowest terms} \\ 1 & \text{otherwise} \end{cases}$