Math 361
Fall, 2003

Test 2 Comments and Solutions

1. Carefully state the following two theorems:

a) the Heine-Borel Theorem

b) the Balzano-Weierstrass Theorem

2. a) Give a careful definition of the following: E is an open set.

b) Give a careful definition of the following: $f : D \to \mathbb{R}$ is continuous at x_0.

c) Give a careful definition of the following: K is compact.

3. Use the **definition** of open to prove that $(0, 1)$ is an open set.

4. Use the **definition** of the limit of a function to prove that $\lim_{x\to 3} \dfrac{x^2 + x - 4}{x - 1} = 4$.

5. Prove **one** of the following. The first is worth more points. Do not use the Heine-Borel Theorem in your proofs.

a) If E is not closed, then E is not compact.

b) $(0, 1)$ is not compact.

Proof of a: Suppose E is not closed. Then there is an accumulation point x_0 of E such that $x_0 \notin E$. Let $B_n = [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$, and let $U_n = \mathbb{R} \setminus B_n$. Each B_n is closed, so each U_n is open. Furthermore, $\bigcap_{n\in\mathbb{J}} B_n = \{x_0\}$, so $\{U_n \mid n \in \mathbb{J}\}$ is an open cover of E. Now for each $n \in \mathbb{J}$:

- $B_n \cap E \neq \emptyset$ (because x_0 is an accumulation point),
- U_n does not cover E (by previous item)
- $U_n \subseteq U_{n+1}$

So no finite subset of $\{U_n \mid n \in \mathbb{J}\}$ covers E.

6. True or False.

a) Indicate (by placing a T or F in the left margin next to each item) which statements are true and which are false. All sets are assumed to be subsets of $\mathbb{R}$. All sequences are assumed to be sequences of real numbers.

i. If E is not open, then E is closed.

False. $[0, 1)$ is an example.

ii. If E is bounded above by 4 and x is an accumulation point of E, then $x \leq 4$.

True. (Use contrapositvie.) If $x > 4$, let $\varepsilon = x - 4$. Then $(x - \varepsilon, x + \varepsilon) \cap E = \emptyset$ because 4 is an upper bound on E. But then x is not an accumulation point.

iii. If $\sup\{a_n : n \in \mathbb{J}\} = 4$, then there is a subsequence of $\{a_n\}_{n=1}^{\infty}$ that converges to 4.

False. Example: $\{3 + \frac{1}{n}\}_{1=\infty}^{\infty}$

iv. If $f(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}$, and $\lim_{x \to 1} f(x) = 0$ and $\lim_{x \to 1} g(x) = 0$, then $\frac{f}{g}$ has no limit at 1.

False. There are many examples. For any differentiable function f, $f'(1)$ is defined by a quotient very much like this (it may not satisfy $f > 0$ and $g > 0$) and from this it is easy to make examples. $f(x) = g(x) = (x-1)^2$ is also an example.

v. If $f : D \to \mathbb{R}$ and $x_0 \in D$, but x_0 is **not** an accumulation point of D, then f is continuous at x_0.

True. In this case, if we choose a small enough δ, x_0 will be the only point x such that $|\, x - x_0 \,| < \delta$. From this it easily follows that f satisfies the definition of continuity at x_0.

vi. If f is continuous on $[1, 4]$, then f is uniformly continuous on $[1, 4]$.

True. This is an instance of the theorem that says continuous functions with compact domains are continuous. [You should not have chosen this to prove, it would take too long for an in class test.]

b) For each item that is false, give a counterexample. (You do not need to give a proof that the counterexample is a counterexample.)

c) Pick **one** item that is true and provide a proof. Feel free to choose the easiest one, subject to the contraint at the top of the page.

7. Let $\{a_n\}_{n=1}^{\infty}$ be a positive, decreasing sequence. Prove that $\{a_n\}_{n=1}^{\infty}$ converges. (A positive sequence is one for which $a_n > 0$ for all $n \in \mathbb{J}$.)

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Name ______________________________

Test 2 – Take Home Portion

You are to work on these problems independently of each other, but you may consult your notes and text. They are due on Tuesday, November 4, at 5 pm.

8. Prove that if $\lim_{x \to x_0} f(x) = L > 0$ and $\lim_{x \to x_0} g(x) = 0$, then $\dfrac{f}{g}$ does not have a limit at x_0.

Proof. Let $M > 0$ be given. Let $\delta > 0$ be such that if $0 < | x - x_0 | < \delta$, then

- $f(x) > L/2$
- $g(x) < \frac{L}{2M}$

Then whenever $0 < | x - x_0 | < \delta$, then $\frac{f(x)}{g(x)} > \frac{L/2}{L/2M} = M$, so $\frac{f}{g}$ is unbounded near 0 and hence can have no limit there.

9. Let's define the limit of a function at infinity as follows:

Definition. Let D be a set of reals with no upper bound, and let $f : D \to \mathbb{R}$. We will say that f has limit L at infinity (and write $\lim_{x \to \infty} f(x) = L$) if and only if for every $\varepsilon > 0$ there is an $M \in \mathbb{R}$ such that whenever $x > M$ and $x \in D$, then $| f(x) - L | < \varepsilon$.

a) Use this definition to prove that $\lim_{x \to \infty} \dfrac{2x^2 + 1}{3x^2 + x} = \dfrac{2}{3}$.

b) Prove that if $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to \infty} g(x) = M$, then $\lim_{x \to \infty} f(x) + g(x) = L + M$.

c) Let f be a function with domain $D = [1, \infty) \to \mathbb{R}$. Let g be a function defined by $g(x) = f(1/x)$; the domain of g is then $(0, 1]$. Prove that f has a limit at infinity if and only if g has a limit at 0.

d) Give a reasonable definition for the following: $\lim_{x \to a} f(x) = \infty$. (Hint: think about other definitions you know.)

10. Define a sequence $\{a_n\}_{n=1}^{\infty}$ recursively as follows:

- $a_1 = 1$,
- for all $n > 1$, $a_n = \dfrac{1}{1 + a_{n-1}}$

a) Write out the first 5 terms of this sequence (as fractions). What do you notice about them?

b) Show that $\{a_n\}_{n=1}^{\infty}$ converges to some real number.

c) Determine the number to which $\{a_n\}_{n=1}^{\infty}$ converges.

11. Let E be a nonempty, compact set of reals. Show that $\sup E \in E$.

12. Some Examples. No proofs required.

a) Give an example of two sequences such that each is a subsequence of the other, but the two sequences are not the same.

b) Find a specific example of a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at *every* point in its domain.

c) Find a specific example of a function $g : \mathbb{R} \to \mathbb{R}$ that is continuous at *exactly one* point in its domain.

d) Find a specific sequence $\{a_n\}_{n=1}^{\infty}$ such that $\{a_n^2\}_{n=1}^{\infty}$ converges, but $\{a_n\}_{n=1}^{\infty}$ does not converge.