Final Exam – Take Home Portion

Instructions.

- You are to work on these problems independently of each other, but you may consult your notes and text. They are due on Thursday, December 18, at 5 pm.

- Be sure to show your work carefully and make it clear when you are using results we have proved in class. (Be sure to check that those results apply in your situation, too.)

- Finally, please check your email occasionally in case there are any updates (corrections of typos, clarifications, extra hints, etc.)

1. Divide and Average. Let \( b \in \mathbb{R} \setminus \{0\} \) and consider the sequence defined by \( a_1 = b \) and \( a_n = \frac{1}{2} \left( a_{n-1} + \frac{b}{a_{n-1}} \right) \) when \( n > 1 \). Do you see why this is referred to as divide and average?
   
   a) Let \( b = 2 \) and compute the first 4 terms of the sequence. Calculate them as fractions and convert the fractions to decimals.
   
   b) Assume for the moment that the sequence converges. What must the limit be?
   
   c) Show that if \( b < 0 \), then the sequence does not converge.
   
   d) Show that if \( b > 0 \), then \( \{a_n\}_{n=1}^{\infty} \) is bounded below.
   
   e) Use induction to show that if \( b \geq 1 \), then \( a_n \geq \sqrt{b} \) for all \( n \in \mathbb{J} \).
      
      If you use the fact that \( b \geq 1 \) in your proof, show where you used it. If you do not use that fact, then say what values of \( b \) your proof is valid for.
   
   f) Show that if \( b \geq 1 \), then \( \{a_n\}_{n=1}^{\infty} \) is decreasing. [Hint: Use part (e).]
   
   g) What can we conclude from parts (d) and (f)?
   
   h) (Extra credit.) Determine (with proof) what happens when \( 0 < b < 1 \).

2. Weird Functions. It is not generally the case that \( f(x + h) = f(x) + h \), although one sometimes sees students making this error in calculus. There are, however, some functions with this property. For this problem, let’s call a function weird if \( f : \mathbb{R} \to \mathbb{R} \) and \( f(x + h) = f(x) + h \) for all \( x \) and all \( h \).

   a) Use the definition of weird to determine which of the following functions are weird: \( f_0(x) = 0 \), \( f_1(x) = x \), \( f_2(x) = x^2 \).
   
   b) Show that if \( f \) is weird, then \( f \) is differentiable on \( \mathbb{R} \), and determine the derivative. [This answer here should agree with your answer in the previous part.]
   
   c) Determine the set of all functions that are weird. [Hint: How is this question different from part (b)?]
3. **True/False.** If true, provide a proof. If false, provide a counterexample.

a) If \( f : D \to \mathbb{R} \) and \( x_0 \in D \) but \( x_0 \) is not an accumulation point of \( D \) (such a point is called an *isolated* point of \( D \)), then \( f \) is continuous at \( x_0 \).

b) If \( \{a_n + b_n\}_{n=1}^{\infty} \) converges to \( K \) and \( \{a_n - b_n\}_{n=1}^{\infty} \) converges to \( L \), then \( \{a_n b_n\}_{n=1}^{\infty} \) converges to \( (K^2 - L^2)/4 \).

c) If \( \{a_n\}_{n=1}^{\infty} \) converges and \( \{a_n b_n\}_{n=1}^{\infty} \) converges, then \( \{b_n\}_{n=1}^{\infty} \) converges.

d) If \( \{a_n\}_{n=1}^{\infty} \) diverges and \( \{a_n b_n\}_{n=1}^{\infty} \) diverges, then \( \{b_n\}_{n=1}^{\infty} \) diverges.

4. **Lipschitzian Functions.** A function \( f : (a, b) \to \mathbb{R} \) is said to be *Lipschitzian* at \( x_0 \) (after a mathematician with the name Lipschitz) if \( x_0 \in (a, b) \) and there is a number \( M \) and a number \( \varepsilon > 0 \) such that

\[
\text{whenever } |x - x_0| < \varepsilon \text{ and } x \in (a, b), \text{ then } |f(x) - f(x_0)| \leq M|x - x_0|.
\]

a) Show that if \( f \) is Lipschitzian, then \( f \) is uniformly continuous.

b) Show that if \( f \) is differentiable at \( x_0 \), then \( f \) is Lipschitzian at \( x_0 \).

c) Give an example of a function that is continuous at a point \( x_0 \) but not Lipschitzian at \( x_0 \).

5. **Taylor’s Theorem.** Let \( f(x) = \ln(1 + x) \).

a) Apply Taylor’s Theorem to \( f \) with \( x_0 = 0 \) to produce a power series. Write out the first four terms of the series and express the series algebraically (using \( \sum \) notation).

b) Find the radius of convergence of this power series. Call it \( r \).

c) It is possible that the power series produced by Taylor’s Theorem converges but does not converge to \( f \). Use the remainder term from Taylor’s Theorem to show that if \( 0 \leq x < r \) (recall \( r \) is the radius of convergence), then the power series converges to \( f(x) \). [Side note: This is true when \(-r < x < 0\) as well, but it is harder to show.]