Sep 9, 2005
The Probability Function

Math 343
Calvin College
Fall 2005
Outline

1. Three Types of Probability
2. Basic Properties of Probability
3. Examples
Definitions

- **Experiment**: a (theoretically infinitely repeatable) procedure with well-defined set of outcomes.
- **Sample Space**: set of all possible outcomes
- **Event**: subset of sample space (possibly empty, possibly all)

Goal of Probability: use numbers to describe the likelihood of events.
Classical Probability

Two assumptions:
- only finitely many outcomes (i.e., sample space is a finite set)
- all outcomes equally likely

The probability of an event is \( \frac{m}{n} \) where \( m = |E| = \text{size of event} \), \( n = |S| = \text{size of sample space} \).

Examples (Simple)
- Coin toss: \( S = \{H, T\} \); \( P(\{H\}) = P(\text{head}) = \frac{1}{2} \)
- Dice roll \( S = \{1, 2, 3, 4, 5, 6\} \); \( P(\{2, 4, 6\}) = P(\text{even}) = \frac{3}{6} \)

Classical Probability is useful, but has limited applicability: What if the assumptions aren’t valid?

Common mistake: apply “equally likely” principle in cases where things are not equally likely.
Empirical Probability

- credited to Richard von Mises (20th c. German)
- Idea: expresses probability as a limit: \( P(E) = \lim_{n \to \infty} \frac{m}{n} \)
  - \( n \) = number of repetitions of experiment
  - \( m \) = number of times event occurs

- Problems with this approach:
  - not clear how to define this limit rigorously
  - idea: approximate probability by evaluating for a large \( n \); but how large is large enough? how accurate is the estimate? what about experiments that are too expensive or impossible to repeat enough times?
Axiomatic Probability

- Kolmogorov, 1933
- focuses on properties (behavior) rather than definition
- consistent with two earlier notions of probability
- minimalist approach
  - list as few axioms as possible
  - from this small list we can derive many other useful properties
Axioms for Probability

1. For any event $E$ defined over any sample space $S$, $P(E) \geq 0$.

2. $P(S) = 1$.

3. If $A$ and $B$ are mutually exclusive events ($A \cap B = \emptyset$), then
   
   \[ P(A \cup B) = P(A) + P(B) \, . \]

4. (for infinite $S$) If $A_1, A_2, \ldots$ is an infinite sequence of mutually exclusive events over a sample space $S$, (i.e., $A_i \cap A_j = \emptyset$ whenever $i \neq j$), then

   \[ P(\bigcup_i A_i) = \sum_i P(A_i) \, . \]

Note: It is easy to check that Classical Probability and von Mises probability have properties 1–3.
Basic Properties of Probability

1. \( P(A^c) = 1 - P(A) \).
2. \( P(\emptyset) = 0 \).
3. If \( A \subseteq B \), then \( P(A) \leq P(B) \).
4. \( P(A) \leq 1 \).
5. If \( A_1, A_2, \ldots, A_n \) are mutually exclusive (\( A_i \cap A_j = \emptyset \) whenever \( i \neq j \)), then
   \[
   P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)
   \]
6. \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \).
Examples

**Example (Warm-up)**
Suppose $P(A) = 0.3$, $P(B) = 0.5$, and $P(A \cup B) = 0.7$. What are $P(A \cap B)$, $P(A^c \cup B^c)$, and $P(A^c \cap B)$?

**Example (2 Dice)**
If we roll two standard dice, what is the probability that we get two different numbers?

**Example (3 Dice)**
If we roll three standard dice, what is the probability that at least one of them is a 6?

**Example (3 Dice again)**
If we roll three standard dice, what is the probability that at least one of them is a even?
More Examples

Example (Two Cards)
Two cards are drawn from a poker deck. What is the probability that the second card has higher rank (denomination) than the first?

Example (Loaded Dice)
Suppose a loaded die is designed so that the number 6 is twice as likely as the other 5 numbers (which are equally likely). What is the probability of rolling a 6? of rolling a 5?

Example (Coin Flips)
If we flip a coin 8 times, what is the probability of getting at least one head and at least one tail?