CHAPTER 3

Section 3.1

1. 

<table>
<thead>
<tr>
<th>S:</th>
<th>FFF</th>
<th>SFF</th>
<th>FSF</th>
<th>FFS</th>
<th>FSS</th>
<th>SFS</th>
<th>SSF</th>
<th>SSS</th>
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</thead>
<tbody>
<tr>
<td>X:</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

2. 

X = 1 if a randomly selected book is non-fiction and X = 0 otherwise
X = 1 if a randomly selected executive is a female and X = 0 otherwise
X = 1 if a randomly selected driver has automobile insurance and X = 0 otherwise

3. 

M = the difference between the large and the smaller outcome with possible values 0, 1, 2, 3, 4, or 5; W = 1 if the sum of the two resulting numbers is even and W = 0 otherwise, a Bernoulli random variable.

4. 

In my perusal of a zip code directory, I found no 00000, nor did I find any zip codes with four zeros, a fact which was not obvious. Thus possible X values are 2, 3, 4, 5 (and not 0 or 1). X = 5 for the outcome 15213, X = 4 for the outcome 44074, and X = 3 for 94322.

5. 

No. In the experiment in which a coin is tossed repeatedly until a H results, let Y = 1 if the experiment terminates with at most 5 tosses and Y = 0 otherwise. The sample space is infinite, yet Y has only two possible values.

6. 

Possible X values are 1, 2, 3, 4, … (all positive integers)

<table>
<thead>
<tr>
<th>Outcome:</th>
<th>RL</th>
<th>AL</th>
<th>RAARL</th>
<th>RRRRL</th>
<th>AARRL</th>
</tr>
</thead>
<tbody>
<tr>
<td>X:</td>
<td>2</td>
<td>2</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>
Chapter 3: Discrete Random Variables and Probability Distributions

7.

a. Possible values are 0, 1, 2, ..., 12; discrete

b. With \( N = \# \) on the list, values are 0, 1, 2, ..., \( N \); discrete

c. Possible values are 1, 2, 3, 4, ...; discrete

d. \( \{ x : 0 < x < \infty \} \) if we assume that a rattlesnake can be arbitrarily short or long; not discrete

e. With \( c = \) amount earned per book sold, possible values are 0, \( c \), 2\( c \), 3\( c \), ..., 10,000\( c \); discrete

f. \( \{ y : 0 < y < 14 \} \) since 0 is the smallest possible pH and 14 is the largest possible pH; not discrete

g. With \( m \) and \( M \) denoting the minimum and maximum possible tension, respectively, possible values are \( \{ x : m < x < M \} \); not discrete

h. Possible values are 3, 6, 9, 12, 15, ... -- i.e. \( 3(1), 3(2), 3(3), 3(4), ... \) giving a first element, etc.; discrete

8. \( Y = 3 : SSS; \quad Y = 4 : FSSS; \quad Y = 5 : FFSSS \), \( SFSSS; \quad Y = 6 : SSFSSS, SFFSSS, FSFSSS, FFFSS; \quad Y = 7 : SSFFS, SFSFFS, SFFFS, FSSFSS, FSFFSS, FFFSS, FFFFFSSS \)

9.

a. Returns to 0 can occur only after an even number of tosses; possible S values are 2, 4, 6, 8, ... (i.e. \( 2(1), 2(2), 2(3), 2(4), ... \)) an infinite sequence, so \( x \) is discrete.

b. Now a return to 0 is possible after any number of tosses greater than 1, so possible values are 2, 3, 4, 5, ... (\( 1+1,1+2, 1+3, 1+4, ... \), an infinite sequence) and \( X \) is discrete

10.

a. \( T = \) total number of pumps in use at both stations. Possible values: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

b. \( X : -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6 \)

c. \( U : 0, 1, 2, 3, 4, 5, 6 \)

d. \( Z : 0, 1, 2 \)
Section 3.2

11.

a.

\[
\begin{array}{c|ccc}
  x & 4 & 6 & 8 \\
  \hline 
  P(x) & .45 & .40 & .15 \\
\end{array}
\]

b.

\[0.0\quad 0.1\quad 0.2\quad 0.3\quad 0.4\quad 0.5\]

\[0.0\quad 4\quad 6\quad 8\]

\[0.0\quad 10\quad 20\quad 30\quad 40\quad 50\]

\[0.0\quad 4\quad 6\quad 7\quad 8\]

\[\text{Relative Frequency}\]

\[\text{x}\]

\[\text{P(x = 6) = .40 + .15 = .55}\]

\[\text{P(x > 6) = .15}\]

c.

\[\text{P(x = 6) = .40 + .15 = .55}\]

\[\text{P(x > 6) = .15}\]

12.

a. In order for the flight to accommodate all the ticketed passengers who show up, no more than 50 can show up. We need \( y = 50 \).

\[\text{P(y = 50) = .05 + .10 + .12 + .14 + .25 + .17 = .83}\]

b. Using the information in a. above, \( P(y > 50) = 1 - P(y = 50) = 1 - .83 = .17 \)

c. For you to get on the flight, at most 49 of the ticketed passengers must show up. \( P(y = 49) = .05 + .10 + .12 + .14 + .25 = .66 \). For the 3\(^{rd}\) person on the standby list, at most 47 of the ticketed passengers must show up. \( P(y = 44) = .05 + .10 + .12 = .27 \)
13.

a. \[ P(X \leq 3) = p(0) + p(1) + p(2) + p(3) = .10 + .15 + .20 + .25 = .70 \]

b. \[ P(X < 3) = P(X \leq 2) = p(0) + p(1) + p(2) = .45 \]

c. \[ P(3 \leq X) = p(3) + p(4) + p(5) + p(6) = .55 \]

d. \[ P(2 \leq X \leq 5) = p(2) + p(3) + p(4) + p(5) = .71 \]

e. The number of lines not in use is 6 – X, so 6 – X = 2 is equivalent to X = 4, 6 – X = 3 to X = 3, and 6 – X = 4 to X = 2. Thus we desire \[ P(2 \leq X \leq 4) = p(2) + p(3) + p(4) = .65 \]

f. 6 – X \geq 4 if 6 – 4 \geq X, i.e. 2 \geq X, or X \leq 2, and \[ P(X \leq 2) = .10 + .15 + .20 = .45 \]

14.

a. \[ \sum_{y=1}^{5} p(y) = K[1 + 2 + 3 + 4 + 5] = 15K = 1 \implies K = \frac{1}{15} \]

b. \[ P(Y \leq 3) = p(1) + p(2) + p(3) = \frac{6}{15} = .4 \]

c. \[ P(2 \leq Y \leq 4) = p(2) + p(3) + p(4) = \frac{9}{15} = .6 \]

d. \[ \sum_{y=1}^{5} \left( \frac{y^2}{50} \right) = \frac{1}{50} [1 + 4 + 9 + 16 + 25] = \frac{55}{50} \neq 1; \text{No} \]

15.

a. (1,2) (1,3) (1,4) (1,5) (2,3) (2,4) (2,5) (3,4) (3,5) (4,5)

b. \[ P(X = 0) = p(0) = P\{ (3,4) (3,5) (4,5)\} = \frac{1}{10} = .3 \]
\[ P(X = 2) = p(2) = P\{ (1,2) \} = \frac{1}{10} = .1 \]
\[ P(X = 1) = p(1) = 1 - [p(0) + p(2)] = .60, \text{and p(x) = 0 if x \neq 0, 1, 2} \]

c. \[ F(0) = P(X \leq 0) = P(X = 0) = .30 \]
\[ F(1) = P(X \leq 1) = P(X = 0 \text{ or } 1) = .90 \]
\[ F(2) = P(X \leq 2) = 1 \]

The c.d.f. is

\[
F(x) = \begin{cases} 
0 & x < 0 \\
.30 & 0 \leq x < 1 \\
.90 & 1 \leq x < 2 \\
1 & 2 \leq x 
\end{cases}
\]
Chapter 3: Discrete Random Variables and Probability Distributions

16. 

a.  

<table>
<thead>
<tr>
<th>$x$</th>
<th>Outcomes</th>
<th>$p(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>FFFF</td>
<td>$(.7)^4$ = .2401</td>
</tr>
<tr>
<td>1</td>
<td>FFFS, FFSF, FSFF, SFFF</td>
<td>$4[(.7)(.3)]^3$ = .4116</td>
</tr>
<tr>
<td>2</td>
<td>FFSS, FSFS, SFFS, FSFF, SFSS, SSFF</td>
<td>$6[(.7)(.3)^2] = .2646$</td>
</tr>
<tr>
<td>3</td>
<td>FSSS, SFSS, SSFS, SSSF</td>
<td>$4[(.7)(.3)]^2 = .0756$</td>
</tr>
<tr>
<td>4</td>
<td>SSSS</td>
<td>$(.3)^4 = .0081$</td>
</tr>
</tbody>
</table>

b.  

![Histogram graph](image)

c. $p(x)$ is largest for $X = 1$

d. $P(X \geq 2) = p(2) + p(3) + p(4) = .2646 + .0756 + .0081 = .3483$

This could also be done using the complement.

17. 

a.  

$P(2) = P(Y = 2) = P(1^{\text{st}} 2 \text{ batteries are acceptable})$  

$= P(\text{AA}) = (.9)(.9) = .81$

b.  

$p(3) = P(Y = 3) = P(\text{UAA or AUA}) = (.1)(.9)^2 + (.1)(.9)^2 = 2[(.1)(.9)^2] = .162$

c. The fifth battery must be an A, and one of the first four must also be an A. Thus, $p(5) = P(\text{AUUUA or UAUUA or UUUAU or UUUAA}) = 4[(.1)(.9)^4] = .00324$

d. $P(Y = y) = p(y) = P(\text{the } y^{\text{th}} \text{ is an A and so is exactly one of the first } y - 1)$  

$= (y - 1)(.1)^{y-2}(.9)^2, \ y = 2, 3, 4, 5, ...$
Chapter 3: Discrete Random Variables and Probability Distributions

18. a. \( p(1) = P(M = 1) = P[(1,1)] = \frac{1}{36} \)
   \[ p(2) = P(M = 2) = P[(1,2) or (2,1) or (2,2)] = \frac{3}{36} \]
   \[ p(3) = P(M = 3) = P[(1,3) or (2,3) or (3,1) or (3,2) or (3,3)] = \frac{5}{36} \]
   Similarly, \( p(4) = \frac{7}{36}, p(5) = \frac{9}{36}, \) and \( p(6) = \frac{11}{36} \)

b. \( F(m) = \)
   \[
   \begin{cases} 
   0 & m < 1, \\
   \frac{1}{36} & 1 \leq m < 2, \\
   \frac{2}{36} & 2 \leq m < 3, \\
   \frac{3}{36} & 3 \leq m < 4, \\
   \frac{4}{36} & 4 \leq m < 5, \\
   \frac{5}{36} & 5 \leq m < 6, \\
   1 & m \geq 6 
   \end{cases}
   \]

19. Let \( A \) denote the type O+ individual (type O positive blood) and B, C, D, the other 3 individuals. Then
   \[ p(1) = P(Y = 1) = P(A \text{ first}) = \frac{1}{4} = .25 \]
   \[ p(2) = P(Y = 2) = P(B, C, \text{ or D first and A next}) = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4} = .25 \]
   \[ p(4) = P(Y = 3) = P(A \text{ last}) = \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{4} = .25 \]
   So \( p(3) = 1 - (.25 + .25 + .25) = .25 \)

20. \( P(0) = P(Y = 0) = P(\text{both arrive on Wed.}) = (.3)(.3) = .09 \)
   \[ P(1) = P(Y = 1) = P[(W, Th) or (Th, W) or (Th, Th)] \]
   \[ = (.3)(.4) + (.4)(.3) + (.4)(.4) = .40 \]
   \[ P(2) = P(Y = 2) = P[(W, F) or (Th, F) or (F, F) or (F, Th) or (Th, Th)] = .32 \]
   \[ P(3) = 1 - [.09 + .40 + .32] = .19 \]
21. The jumps in \( F(x) \) occur at \( x = 0, 1, 2, 3, 4, 5, \) and 6, so we first calculate \( F( ) \) at each of these values:

\[
F(0) = P(X \leq 0) = P(X = 0) = .10 \\
F(1) = P(X \leq 1) = p(0) + p(1) = .25 \\
F(2) = P(X \leq 2) = p(0) + p(1) + p(2) = .45 \\
F(3) = .70, F(4) = .90, F(5) = .96, \text{ and } F(6) = 1.
\]

The c.d.f. is:

\[
F(x) = \begin{cases} 
0.00 & x < 0 \\
0.10 & 0 \leq x < 1 \\
0.25 & 1 \leq x < 2 \\
0.45 & 2 \leq x < 3 \\
0.70 & 3 \leq x < 4 \\
0.90 & 4 \leq x < 5 \\
0.96 & 5 \leq x < 6 \\
1.00 & 6 \leq x
\end{cases}
\]

Then \( P(X \leq 3) = F(3) = .70, P(X < 3) = P(X \leq 2) = F(2) = .45, \)

\( P(3 \leq X) = 1 - P(X \leq 2) = 1 - F(2) = 1 - .45 = .55, \)

and \( P(2 \leq X \leq 5) = F(5) - F(1) = .96 - .25 = .71 \)

22.

a. \( P(X = 2) = .39 - .19 = .20 \)
b. \( P(X > 3) = 1 - .67 = .33 \)
c. \( P(2 \leq X \leq 5) = .92 - .19 = .73 \)
d. \( P(2 < X < 5) = .92 - .39 = .53 \)

23.

a. Possible \( X \) values are those values at which \( F(x) \) jumps, and the probability of any particular value is the size of the jump at that value. Thus we have:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>.30</td>
<td>.10</td>
<td>.05</td>
<td>.15</td>
<td>.40</td>
</tr>
</tbody>
</table>

b. \( P(3 \leq X \leq 6) = F(6) - F(3-) = .60 - .30 = .30 \)

\( P(4 \leq X) = 1 - P(X < 4) = 1 - F(4-) = 1 - .40 = .60 \)

24. \( P(0) = P(Y = 0) = P(B \text{ first}) = p \)

\( P(1) = P(Y = 1) = P(G \text{ first, then } B) = P(GB) = (1 - p)p \)

\( P(2) = P(Y = 2) = P(GGB) = (1 - p)^2p \)

Continuing, \( p(y) = P(Y=y) = P(y G's \text{ and then a } B) = (1 - p)^y p \text{ for } y = 0, 1, 2, 3, \ldots \)
Chapter 3: Discrete Random Variables and Probability Distributions

25.  
   a. Possible X values are 1, 2, 3, …
      
      \[ P(1) = P(X = 1) = P(\text{return home after just one visit}) = \frac{1}{3} \]
      
      \[ P(2) = P(X = 2) = P(\text{second visit and then return home}) = \frac{2}{3} \cdot \frac{1}{3} \]
      
      \[ P(3) = P(X = 3) = P(\text{three visits and then return home}) = \left( \frac{2}{3} \right)^2 \cdot \frac{1}{3} \]
      
      In general \( p(x) = \left( \frac{2}{3} \right)^{x-1} \left( \frac{1}{3} \right) \) for \( x = 1, 2, 3, \ldots \)
   
   b. The number of straight line segments is \( Y = 1 + X \) (since the last segment traversed returns Alvie to O), so as in a, \( p(y) = \left( \frac{2}{3} \right)^{y-2} \left( \frac{1}{3} \right) \) for \( y = 2, 3, \ldots \)
   
   c. Possible Z values are 0, 1, 2, 3, …
      
      \[ p(0) = P(\text{male first and then home}) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} \]
      
      \[ p(1) = P(\text{exactly one visit to a female}) = P(\text{female 1st, then home}) + P(\text{F, M, home}) + P(\text{M, F, home}) + P(\text{M, F, M, home}) \]
      
      \[ = \left( \frac{1}{2} \right) \left( \frac{1}{3} \right) + \left( \frac{1}{2} \right) \left( \frac{2}{3} \right) \left( \frac{1}{3} \right) + \left( \frac{1}{2} \right) \left( \frac{2}{3} \right) \left( \frac{1}{3} \right) + \left( \frac{1}{2} \right) \left( \frac{1}{3} \right) \left( \frac{2}{3} \right) \left( \frac{1}{3} \right) \]
      
      where the first term corresponds to initially visiting a female and the second term corresponds to initially visiting a male. Similarly,
      
      \[ p(2) = \left( \frac{1}{2} \right) \left( \frac{2}{3} \right) \left( \frac{1}{3} \right) + \left( \frac{1}{2} \right) \left( \frac{1}{3} \right) \left( \frac{2}{3} \right) \left( \frac{1}{3} \right) \]
      
      In general, \( p(z) = \left( \frac{1}{2} \right) \left( \frac{2}{3} \right)^{z-2} \left( \frac{1}{3} \right) + \left( \frac{1}{2} \right) \left( \frac{1}{3} \right)^z \left( \frac{1}{3} \right) \) for \( z = 1, 2, 3, \ldots \)

26.  
   a. The sample space consists of all possible permutations of the four numbers 1, 2, 3, 4:
      
      | outcome | y value | outcome | y value | outcome | y value |
      |---------|--------|---------|--------|---------|--------|
      | 1234    | 4      | 2314    | 1      | 3412    | 0      |
      | 1243    | 2      | 2341    | 0      | 3421    | 0      |
      | 1324    | 2      | 2413    | 0      | 4132    | 1      |
      | 1342    | 1      | 2431    | 1      | 4123    | 0      |
      | 1423    | 1      | 3124    | 1      | 4213    | 1      |
      | 1432    | 2      | 3142    | 0      | 4231    | 2      |
      | 2134    | 2      | 3214    | 2      | 4312    | 0      |
      | 2143    | 0      | 3241    | 1      | 4321    | 0      |

   b. Thus \( P(0) = P(Y = 0) = \frac{9}{24} \), \( P(1) = P(Y = 1) = \frac{8}{24} \), \( P(2) = P(Y = 2) = \frac{6}{24} \), \( P(3) = P(Y = 3) = 0 \), \( P(3) = P(Y = 3) = \frac{1}{24} \).
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27. If \( x_1 < x_2 \), \( F(x_2) = P(X \leq x_2) = P(\{X \leq x_1\} \cup \{x_1 < X \leq x_2\}) = P(X \leq x_1) + P(x_1 < X \leq x_2) \geq P(X \leq x_1) = F(x_1). \)

\[ F(x_1) = F(x_2) \text{ when } P(x_1 < X \leq x_2) = 0. \]

Section 3.3

28.

a. \( E(X) = \sum_{x=0}^{4} x \cdot p(x) = (0)(.08) + (1)(.15) + (2)(.45) + (3)(.27) + (4)(.05) = 2.06 \)

b. \( V(X) = \sum_{x=0}^{4} (x - 2.06)^2 \cdot p(x) = (0 - 2.06)^2(.08) + \ldots + (4 - 2.06)^2(.05) \)

\[ = .339488+.168540+.001620+.238572+.188180 = .9364 \]

c. \( \sigma_x = \sqrt{.9364} = .9677 \)

d. \( V(X) = \left[ \sum_{x=0}^{4} x^2 \cdot p(x) \right] - (2.06)^2 = 5.1800 - 4.2436 = .9364 \)

29.

a. \( E(Y) = \sum_{x=0}^{4} y \cdot p(y) = (0)(.60) + (1)(.25) + (2)(.10) + (3)(.05) = .60 \)

b. \( E(100Y^2) = \sum_{x=0}^{4} 100y^2 \cdot p(y) = (0)(.60) + (100)(.25) + (400)(.10) + (900)(.05) = 110 \)

30. \( E(Y) = .60; \)

\( E(Y^2) = 1.1 \)

\( V(Y) = E(Y^2) - [E(Y)]^2 = 1.1 - (.60)^2 = .74 \)

\( \sigma_y = \sqrt{.74} = .8602 \)

\( E(Y) \pm \sigma = .60 \pm .8602 = (-.2602, 1.4602) \) or (0, 1).

\( P(Y = 0) + P(Y = 1) = .85 \)
31.  
   a.  \( E(X) = (13.5)(.2) + (15.9)(.5) + (19.1)(.3) = 16.38 \),  
       \( E(X^2) = (13.5)^2(.2) + (15.9)^2(.5) + (19.1)^2(.3) = 272.298 \),  
       \( V(X) = 272.298 - (16.38)^2 = 3.9936 \)  
   b.  \( E(25X - 8.5) = 25 E(X) - 8.5 = (25)(16.38) - 8.5 = 401 \)  
   c.  \( V(25X - 8.5) = V(25X) = (25)^2 V(X) = (625)(3.9936) = 2496 \)  
   d.  \( E[h(X)] = E[X - .01X^2] = E(X) - .01E(X^2) = 16.38 - 2.72 = 13.66 \)

32.  
   a.  \( E(X^2) = \sum_{x=0}^{\infty} x^2 p(x) = (0^2)(1-p) + (1^2)p = (1)p = p \)  
   b.  \( V(X) = E(X^2) - [E(X)]^2 = p - p^2 = p(1-p) \)  
   c.  \( E(X^3) = (0^3)(1-p) + (1^3)p = p \)

33.  
   \( E(X) = \sum_{x=1}^{\infty} x \cdot p(x) = \sum_{x=1}^{\infty} x \cdot \frac{c}{x^3} = c \sum_{x=1}^{\infty} \frac{1}{x^2} \), but it is a well-known result from the theory of infinite series that \( \sum_{x=1}^{\infty} \frac{1}{x^2} < \infty \), so \( E(X) \) is finite.

34.  
   Let \( h(X) \) denote the net revenue (sales revenue – order cost) as a function of \( X \). Then \( h_3(X) \) and \( h_4(X) \) are the net revenue for 3 and 4 copies purchased, respectively. For \( x = 1 \) or 2, \( h_3(X) = 2x - 3 \), but at \( x = 3,4,5,6 \) the revenue plateaus. Following similar reasoning, \( h_4(X) = 2x - 4 \) for \( x=1,2,3 \), but plateaus at 4 for \( x = 4,5,6 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_3(x) )</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( h_4(x) )</td>
<td>-2</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>( p(x) )</td>
<td>( \frac{1}{15} )</td>
<td>( \frac{7}{15} )</td>
<td>( \frac{1}{15} )</td>
<td>( \frac{3}{15} )</td>
<td>( \frac{4}{15} )</td>
<td>( \frac{3}{15} )</td>
</tr>
</tbody>
</table>

\[ E[h_3(X)] = \sum_{x=1}^{6} h_3(x) \cdot p(x) = (-1)(\frac{1}{15}) + \ldots + (3)(\frac{3}{15}) = 2.4667 \]

Similarly, \( E[h_4(X)] = \sum_{x=1}^{6} h_4(x) \cdot p(x) = (-2)(\frac{1}{15}) + \ldots + (4)(\frac{2}{15}) = 2.6667 \)

Ordering 4 copies gives slightly higher revenue, on the average.
35. 

<table>
<thead>
<tr>
<th>( x )</th>
<th>( P(x) )</th>
<th>.8</th>
<th>.1</th>
<th>.08</th>
<th>.02</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>0</td>
<td>1,000</td>
<td>5,000</td>
<td>10,000</td>
<td></td>
</tr>
<tr>
<td>( H(x) )</td>
<td>0</td>
<td>500</td>
<td>4,500</td>
<td>9,500</td>
<td></td>
</tr>
</tbody>
</table>

\[ E[h(X)] = 600. \text{ Premium should be$100 plus expected value of damage minus deductible or$700.} \]

36. \[
E(X) = \sum_{x=1}^{n} x \cdot \left( \frac{1}{n} \right) = \left( \frac{1}{n} \right) \sum_{x=1}^{n} x = \frac{1}{n} \left[ \frac{n(n+1)}{2} \right] = \frac{n+1}{2}
\]

\[
E(X^2) = \sum_{x=1}^{n} x^2 \cdot \left( \frac{1}{n} \right) = \left( \frac{1}{n} \right) \sum_{x=1}^{n} x^2 = \frac{1}{n} \left[ \frac{n(n+1)(2n+1)}{6} \right] = \frac{(n+1)(2n+1)}{6}
\]

\[
So \ V(X) = \frac{(n+1)(2n+1)}{6} - \left( \frac{n+1}{2} \right)^2 = \frac{n^2 - 1}{12}
\]

37. \[
E[h(X)] = E\left( \frac{1}{X} \right) = \sum_{x=1}^{6} \frac{1}{x} \cdot p(x) = \frac{1}{6} \sum_{x=1}^{6} \frac{1}{x} = .408, \text{ whereas } \frac{1}{3.5} = .286, \text{ so you expect to win more if you gamble.}
\]

38. \[
E(X) = \sum_{x=1}^{4} x \cdot p(x) = 2.3, \text{ E}(X^2) = 6.1, \text{ so } V(X) = 6.1 - (2.3)^2 = .81
\]

Each lot weighs 5 lbs, so weight left = 100 – 5x.
Thus the expected weight left is 100 – 5E(X) = 88.5,
and the variance of the weight left is
\[ V(100 - 5X) = V(-5X) = 25V(X) = 20.25. \]

39. 

a. The line graph of the p.m.f. of \(-X\) is just the line graph of the p.m.f. of \(X\) reflected about zero, but both have the same degree of spread about their respective means, suggesting \(V(-X) = V(X)\).

b. With \(a = -1, b = 0\), \(V(aX + b) = V(-X) = a^2V(X)\).

40. \[
V(aX + b) = \sum_{x} [aX + b - E(aX + b)]^2 \cdot p(x) = \sum_{x} [aX + b - (a\mu + b)]^2 \cdot p(x)
\]

\[
= \sum_{x} [aX - (a\mu)]^2 \cdot p(x) = a^2 \sum_{x} [X - \mu]^2 \cdot p(x) = a^2V(X).
\]
41.  
   a. \( E[X(X-1)] = E(X^2) - E(X) \) \Rightarrow E(X^2) = E[X(X-1)] + E(X) = 32.5
   
   b. \( V(X) = E(X^2) - (E(X))^2 = 32.5 - (5)^2 = 7.5 \)
   
   c. \( V(X) = E[X(X-1)] + E(X) - [E(X)]^2 \)

42.  
   With \( a = 1 \) and \( b = c \), \( E(X - c) = E(aX + b) = aE(X) + b = E(X) - c \). When \( c = \mu \), \( E(X - \mu) = E(X) - \mu = \mu - \mu = 0 \), so the expected deviation from the mean is zero.

43.  
   a. 
   
   \[
   \begin{array}{c|cccccc}
   k & 2 & 3 & 4 & 5 & 10 \\
   \hline
   \frac{1}{k^2} & .25 & .11 & .06 & .04 & .01 \\
   \end{array}
   \]

   b. \( \mu = \sum_{x=0}^{6} x \cdot p(x) = 2.64 \), \( \sigma^2 = \left[ \sum_{x=0}^{6} x^2 \cdot p(x) \right] - \mu^2 = 2.37, \sigma = 1.54 \)
   
   Thus \( \mu - 2\sigma = -.44 \), and \( \mu + 2\sigma = 5.72 \),
   so \( P(|X-\mu| \geq 2\sigma) = P(X \text{ is at least 2 s.d.'s from } \mu) = P(X = 6) = .04 \).

   Chebyshev’s bound of .025 is much too conservative. For \( K = 3,4,5, \) and \( 10 \), \( P(|X-\mu| \geq k\sigma) = 0 \), here again pointing to the very conservative nature of the bound \( \frac{1}{k^2} \).

   c. \( \mu = 0 \) and \( \sigma = \frac{1}{3} \), so \( P(|X-\mu| \geq 3\sigma) = P(|X| \geq 1) = P(X = -1 \text{ or } +1) = \frac{1}{18} + \frac{1}{18} = \frac{1}{9} \), identical to the upper bound.

   d. Let \( p(-1) = \frac{1}{30} \), \( p(1) = \frac{1}{30} \), \( p(0) = \frac{24}{25} \).
Section 3.4

44. 
   a. \( b(3;8,.6) = \binom{8}{3} (.6)^3 (.4)^5 = (56)(.00221184) = .124 \)

   b. \( b(5;8,.6) = \binom{8}{5} (.6)^5 (.4)^3 = (56)(.00497664) = .279 \)

   c. \( P(3 \leq X \leq 5) = b(3;8,.6) + b(4;8,.6) + b(5;8,.6) = .635 \)

   d. \( P(1 \leq X) = 1 - P(X = 0) = 1 - \binom{12}{0} (.1)^0 (.9)^{12} = 1 - (.9)^{12} = .718 \)

45. 
   a. \( B(4;10,.3) = .850 \)

   b. \( b(4;10,.3) = B(4;10,.3) - B(3;10,.3) = .200 \)

   c. \( b(6;10,.7) = B(6;10,.7) - B(5;10,.7) = .200 \)

   d. \( P(2 \leq X \leq 4) = B(4;10,.3) - B(1;10,.3) = .701 \)

   e. \( P(2 < X) = 1 - P(X \leq 1) = 1 - B(1;10,.3) = .851 \)

   f. \( P(X \leq 1) = B(1;10,.7) = .0000 \)

   g. \( P(2 < X < 6) = P(3 \leq X \leq 5) = B(5;10,.3) - B(2;10,.3) = .570 \)

46. \( X \sim \text{Bin}(25, .05) \)
   a. \( P(X \leq 2) = B(2;25,.05) = .873 \)

   b. \( P(X \geq 5) = 1 - P(X \leq 4) = 1 - B(4;25,.05) = .1 - .993 = .007 \)

   c. \( P(1 \leq X \leq 4) = P(X \leq 4) - P(X \leq 0) = .993 - .277 = .716 \)

   d. \( P(X = 0) = P(X \leq 0) = .277 \)

   e. \( E(X) = np = (25)(.05) = 1.25 \)

   \( V(X) = np(1 - p) = (25)(.05)(.95) = 1.1875 \)

   \( \sigma_x = 1.0897 \)
47. $X \sim \text{Bin}(6, .10)$

a. $P(X = 1) = \binom{n}{x} p^x (1 - p)^{n-x} = \binom{6}{1} (.1)^1 (.9)^5 = .3543$

b. $P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$.

From a, we know $P(X = 1) = .3543$, and $P(X = 0) = \binom{6}{0} (.1)^0 (.9)^6 = .5314$.

Hence $P(X \geq 2) = 1 - (.3543 + .5314) = .1143$

c. Either 4 or 5 goblets must be selected

i) Select 4 goblets with zero defects: $P(X = 0) = \binom{4}{0} (.1)^0 (.9)^4 = .6561$.

ii) Select 4 goblets, one of which has a defect, and the 5\textsuperscript{th} is good:

\[ \binom{4}{1} (.1)^1 (.9)^3 \times .9 = .26244 \]

So the desired probability is $.6561 + .26244 = .91854$

48. Let $S = \text{comes to a complete stop}$, so $p = .25$, $n = 20$

a. $P(X \leq 6) = B(6;20,.25) = .786$

b. $P(X = 6) = b(6;20,.20) = B(6;20,.25) - B(5;20,.25) = .786 - .617 = .169$

c. $P(X \geq 6) = 1 - P(X \leq 5) = 1 - B(5;20,.25) = 1 - .617 = .383$

d. $E(X) = (20)(.25) = 5$. We expect 5 of the next 20 to stop.

49. Let $S = \text{has at least one citation}$. Then $p = .4$, $n = 15$

a. If at least 10 have no citations (Failure), then at most 5 have had at least one (Success):

\[ P(X \leq 5) = B(5;15,.40) = .403 \]

b. $P(X \leq 7) = B(7;15,.40) = .787$

c. $P(5 \leq X \leq 10) = P(X \leq 10) - P(X \leq 4) = .991 - .217 = .774$
50. \(\text{X} \sim \text{Bin}(10, .60)\)
   a. \(P(\text{X} \geq 6) = 1 - P(\text{X} \leq 5) = 1 - \text{B}(5;20,.60) = 1 - .367 = .633\)
   b. \(E(\text{X}) = np = (10)(.6) = 6\); \(V(\text{X}) = np(1 - p) = (10)(.6)(.4) = 2.4\);
      \(\sigma_x = 1.55\)
      \(E(\text{X}) \pm \sigma_x = (4.45, 7.55)\).
      We desire \(P(5 \leq \text{X} \leq 7) = P(\text{X} \leq 7) - P(\text{X} \leq 4) = .833 - .166 = .667\)
   c. \(P(3 \leq \text{X} \leq 7) = P(\text{X} \leq 7) - P(\text{X} \leq 2) = .833 - .012 = .821\)

51. Let \(S\) represent a telephone that is submitted for service while under warranty and must be
    replaced. Then \(p = P(\text{S}) = P(\text{replaced} | \text{submitted})P(\text{submitted}) = (.40)(.20) = .08.\) Thus \(\text{X}\),
    the number among the company’s 10 phones that must be replaced, has a binomial distribution with \(n = 10, p = .08\),
    so \(p(2) = P(\text{X} = 2) = \binom{10}{2}(.08)^2 (.92)^8 = .1478\)

52. \(\text{X} \sim \text{Bin}(25, .02)\)
   a. \(P(\text{X} = 1) = 25(.02)^2 (.98)^{24} = .308\)
   b. \(P(\text{X} = 1) = 1 - P(\text{X} = 0) = 1 - (.98)^{25} = 1 - .603 = .397\)
   c. \(P(\text{X} = 2) = 1 - P(\text{X} = 1) = 1 - [.308 + .397]\)
   d. \(\bar{x} = 25(.02) = .5; \ \sigma = \sqrt{npq} = \sqrt{25(.02)(.98)} = \sqrt{.49} = .7\)
      \(\bar{x} + 2\sigma = .5 + 1.4 = 1.9\) So \(P(0 = \text{X} = 1.9 = P(\text{X} = 1) = .705\)
   e. \(\frac{.5(4.5) + 24.5(3)}{25} = 3.03 \text{ hours}\)

53. \(X = \) the number of flashlights that work.

    Let event \(B = \{\) battery has acceptable voltage\}\).
    Then \(P(\text{flashlight works}) = P(\text{both batteries work}) = P(B)P(B) = (.9)(.9) = .81\) We must
    assume that the batteries’ voltage levels are independent.
    \(X \sim \text{Bin}(10, .81). \ P(\text{X} = 9) = P(\text{X} = 9) + P(\text{X} = 10)\)
    \(\left(\binom{10}{9}(.81)^9(.19) + \binom{10}{10}(.81)^{10}\right) = .285 + .122 = .407\)
Let $p$ denote the actual proportion of defectives in the batch, and $X$ denote the number of defectives in the sample.

**a.** $P$(the batch is accepted) = $P(X \leq 2) = B(2;10,p)$

<table>
<thead>
<tr>
<th>$p$</th>
<th>.01</th>
<th>.05</th>
<th>.10</th>
<th>.20</th>
<th>.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(\text{accept})$</td>
<td>1.00</td>
<td>.988</td>
<td>.930</td>
<td>.678</td>
<td>.526</td>
</tr>
</tbody>
</table>

**b.**

**c.** $P$(the batch is accepted) = $P(X \leq 1) = B(1;10,p)$

<table>
<thead>
<tr>
<th>$p$</th>
<th>.01</th>
<th>.05</th>
<th>.10</th>
<th>.20</th>
<th>.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(\text{accept})$</td>
<td>.996</td>
<td>.914</td>
<td>.736</td>
<td>.376</td>
<td>.244</td>
</tr>
</tbody>
</table>

**d.** $P$(the batch is accepted) = $P(X \leq 2) = B(2;15,p)$

<table>
<thead>
<tr>
<th>$p$</th>
<th>.01</th>
<th>.05</th>
<th>.10</th>
<th>.20</th>
<th>.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(\text{accept})$</td>
<td>1.00</td>
<td>.964</td>
<td>.816</td>
<td>.398</td>
<td>.236</td>
</tr>
</tbody>
</table>

**e.** We want a plan for which $P(\text{accept})$ is high for $p \leq .1$ and low for $p > .1$.

The plan in **d** seems most satisfactory in these respects.
55. 
   a. \( P(\text{rejecting claim when } p = .8) = B(15; 25, .8) = .017 \)
   
   b. \( P(\text{not rejecting claim when } p = .7) = P(X \geq 16 \text{ when } p = .7) = 1 - B(15; 25, .7) = 1 - .189 = .811 \); for \( p = .6 \), this probability is \( 1 - B(15; 25, .6) = 1 - .575 = .425 \).
   
   c. The probability of rejecting the claim when \( p = .8 \) becomes \( B(14; 25, .8) = .006 \), smaller than in \( a \) above. However, the probabilities of \( b \) above increase to .902 and .586, respectively.

56. \( h(x) = 1 \cdot X + 2.25(25 - X) = 62.5 - 1.5X \), so \( E(h(X)) = 62.5 - 1.5E(x) = 62.5 - 1.5np - 62.5 - (1.5)(25)(.6) = \$40.00 \)

57. If topic A is chosen, when \( n = 2 \), \( P(\text{at least half received}) = P(X \geq 1) = 1 - P(X = 0) = 1 - (.1)^2 = .99 \)
   
   If B is chosen, when \( n = 4 \), \( P(\text{at least half received}) = P(X \geq 2) = 1 - P(X \leq 1) = 1 - (0.1)^4 - 4(.1)^3(.9) = .9963 \)
   
   Thus topic B should be chosen.
   
   If \( p = .5 \), the probabilities are .75 for A and .6875 for B, so now A should be chosen.

58. 
   a. \( np(1 - p) = 0 \) if either \( p = 0 \) (whence every trial is a failure, so there is no variability in \( X \)) or if \( p = 1 \) (whence every trial is a success and again there is no variability in \( X \))
   
   b. \( \frac{d}{dp}[np(1 - p)] = n[1 - 2p = 0 \quad \Rightarrow \quad p = .5 \), which is easily seen to correspond to a maximum value of \( V(X) \).

59. 
   a. \( b(x; n, 1 - p) = \binom{n}{x}(1 - p)^x(p)^{n-x} = \binom{n}{n-x}(p)^x(1 - p)^{n-x} = b(n-x; n, p) \)

   Alternatively, \( P(\text{x S’s when } P(S) = 1 - p) = P(\text{n-x F’s when } P(F) = p), \) since the two events are identical), but the labels S and F are arbitrary so can be interchanged (if \( P(S) \) and \( P(F) \) are also interchanged), yielding \( P(\text{n-x S’s when } P(S) = 1 - p) \) as desired.

   b. \( B(x;n,1 - p) = P(\text{at most x S’s when } P(S) = 1 - p) = P(\text{at least n-x F’s when } P(F) = p) = P(\text{at least n-x S’s when } P(S) = p) = 1 - P(\text{at most n-x-1 S’s when } P(S) = p) = 1 - B(n-x-1;n, p) \)

   c. Whenever \( p > .5 \), \( (1 - p) < .5 \) so probabilities involving \( X \) can be calculated using the results \( a \) and \( b \) in combination with tables giving probabilities only for \( p \leq .5 \)
Chapter 3: Discrete Random Variables and Probability Distributions

60. Proof of E(X) = np:

\[ E(X) = \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \]

\[ = \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} \]

\[ = np \left\{ \sum_{y=0}^{n-1} \frac{n-1}{y} p^y (1-p)^{n-1-y} \right\} \]

The expression in braces is the sum over all possible values \( y = 0, 1, 2, \ldots, n-1 \) of a binomial p.m.f. based on \( n-1 \) trials, so equals 1, leaving only np, as desired.

61. a. Although there are three payment methods, we are only concerned with \( S = \) uses a debit card and \( F = \) does not use a debit card. Thus we can use the binomial distribution. So \( n = 100 \) and \( p = .5 \). E(X) = np = 100(.5) = 50, and V(X) = 25.

b. With \( S = \) doesn’t pay with cash, \( n = 100 \) and \( p = .7 \), E(X) = np = 100(.7) = 70, and V(X) = 21.

62. a. Let \( X = \) the number with reservations who show, a binomial r.v. with \( n = 6 \) and \( p = .8 \). The desired probability is

\[ P(X = 5 \text{ or } 6) = \binom{6}{5}(.8)^5(.2) + \binom{6}{6}(.8)^6 = .3932 + .2621 = .6553 \]

b. Let \( h(X) = \) the number of available spaces. Then

\[ \begin{array}{c|c|c|c|c|c|c|}
\text{When x is:} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\text{H(x) is:} & 4 & 3 & 2 & 1 & 0 & 0 & 0 \\
\end{array} \]

\[ E[h(X)] = \sum_{x=0}^{6} h(x) \cdot b(x;6,.8) = 4(.000) + 3(.002) = .0172 \]

\[ P(X = 0) = b(0;3,.8)(.1) + b(0;4,.8)(.2) + b(0;5,.8)(.3) + b(0;6,.8)(.4) = .0080(.1) + .0016(.2) + .0003(.3) + .0001(.4) = .0113 \]

\[ P(X = 1) = b(1;3,.8)(.1) + \ldots + b(1;6,.8)(.4) = .0172 \]

\[ P(X = 2) = .0906, \quad P(X = 3) = .2273, \]

\[ P(X = 4) = 1 - [ .0113 + \ldots + .2273 ] = .6636 \]
63. When \( p = .5 \), \( \mu = 10 \) and \( \sigma = 2.236 \), so \( 2\sigma = 4.472 \) and \( 3\sigma = 6.708 \).

The inequality \(|X - 10| \geq 4.472\) is satisfied if either \( X \leq 5 \) or \( X \geq 15 \), or \( P(|X - \mu| \geq 2\sigma) = P(X \leq 5 \text{ or } X \geq 15) = .021 + .021 = .042 \).

In the case \( p = .75 \), \( \mu = 15 \) and \( \sigma = 1.937 \), so \( 2\sigma = 3.874 \) and \( 3\sigma = 5.811 \). \( P(|X - 15| \geq 3.874) = P(X \leq 11 \text{ or } X \geq 19) = .041 + .024 = .065 \), whereas \( P(|X - 15| \geq 5.811) = P(X \leq 9) = .004 \). All these probabilities are considerably less than the upper bounds .25 (for \( k = 2 \)) and .11 (for \( k = 3 \)) given by Chebyshev.

Section 3.5

64.

a. \( X \sim \text{Hypergeometric } N=15, n=5, M=6 \)

b. \( P(X=2) = \binom{6}{2} \frac{9}{15} \binom{9}{3} \frac{15}{5} = \frac{840}{3003} = .280 \)

\[ P(X=2) = P(X=0) + P(X=1) + P(X=2) \]

\[ = \binom{9}{5} \frac{6}{15} \binom{15}{4} \frac{9}{15} + \frac{840}{3003} = 126 + 756 + 840 = \frac{1722}{3003} = .573 \]

\[ P(X=2) = 1 - P(X=1) = 1 - [P(X=0) + P(X=1)] = 1 - \frac{126 + 756}{3003} = .706 \]

c. \( E(X) = \sum \frac{6}{15} = 2 \); \( V(X) = \left( \frac{15 - 5}{14} \right) \cdot 5 \cdot \left( \frac{6}{15} \right) \left( 1 - \frac{6}{15} \right) = .857 \)

\( \sigma = \sqrt{V(X)} = .926 \)
65. \(X \sim h(x; 6, 12, 7)\)

a. \(P(X=5) = \binom{7}{5} \binom{1}{1} \binom{12}{6} \binom{6}{5} = \frac{105}{924} = .114\)

b. \(P(X=4) = 1 - P(X=5) = 1 - [P(X=5) + P(X=6)] =\)

\[
1 - \left( \binom{7}{5} \binom{1}{1} \binom{12}{6} \binom{6}{5} + \binom{6}{1} \binom{1}{1} \binom{12}{6} \binom{6}{6} \right) = 1 - \frac{105 + 7}{924} = 1 - \frac{1.121}{924} = .879
\]

c. \(E(X) = \left( \frac{11}{12} \cdot 7 \right) = 3.5; \ \sigma = \sqrt{\left( \frac{11}{12} \right) \left( \frac{7}{12} \right) \left( \frac{5}{12} \right)} = \sqrt{.795} = .892\)

\(P(X > 3.5 + .892) = P(X > 4.392) = P(X=5) = .121\) (see part b)

d. We can approximate the hypergeometric distribution with the binomial if the population size and the number of successes are large: \(h(x;15,40,400)\) approaches \(b(x;15,.10)\). So \(P(X=5) \approx B(5; 15, .10)\) from the binomial tables = .998

66.

a. \(P(X = 10) = h(10;15,30,50) = \binom{30}{10} \binom{20}{5} = \binom{50}{15} = .2070\)

b. \(P(X \geq 10) = h(10;15,30,50) + h(11;15,30,50) + \ldots + h(15;15,30,50) = .2070 + .1176 + .0438 + .0101 + .0013 + .0001 = .3799\)

c. \(P(\text{at least 10 from the same class}) = P(\text{at least 10 from second class [answer from b]}) + P(\text{at least 10 from first class}). \text{ But "at least 10 from 1st class" is the same as "at most 5 from the second" or } P(X \leq 5).\)

\(P(X \leq 5) = h(0;15,30,50) + h(1;15,30,50) + \ldots + h(5;15,30,50) = 11697 + .002045 + .000227 + .000150 + .00001 + .000001 = .01412\)

So the desired probability = \(P(x \geq 10) + P(X \leq 5) = .3799 + .01412 = .39402\)
Chapter 3: Discrete Random Variables and Probability Distributions

d. \[ E(X) = n \cdot \frac{M}{N} = 15 \cdot \frac{30}{50} = 9 \]
\[ V(X) = \left( \frac{35}{49} \right) \cdot 9 \cdot \left( 1 - \frac{30}{50} \right) = 2.5714 \]
\[ \sigma_x = 1.6036 \]

e. Let \( Y = 15 - X \). Then \( E(Y) = 15 - E(X) = 15 - 9 = 6 \)
\[ V(Y) = V(15 - X) - V(X) = 2.5714, \text{ so } \sigma_Y = 1.6036 \]

67.
a. Possible values of \( X \) are 5, 6, 7, 8, 9, 10. (In order to have less than 5 of the granite, there would have to be more than 10 of the basaltic).
\[ P(X = 5) = h(5; 15, 10, 20) = \frac{\binom{10}{5} \binom{10}{10}}{\binom{20}{15}} = .0163. \]
Following the same pattern for the other values, we arrive at the pmf, in table form below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>.0163</td>
<td>.1354</td>
<td>.3483</td>
<td>.3483</td>
<td>.1354</td>
<td>.0163</td>
</tr>
</tbody>
</table>

b. \( P(\text{all 10 of one kind or the other}) = P(X = 5) + P(X = 10) = .0163 + .0163 = .0326 \)

c. \[ E(X) = n \cdot \frac{M}{N} = 15 \cdot \frac{10}{20} = 7.5 ; V(X) = \left( \frac{5}{19} \right) \cdot 7.5 \left( 1 - \frac{10}{20} \right) = .9868 ; \]
\[ \sigma_x = .9934 \]
\[ \mu \pm \sigma = 7.5 \pm .9934 = (6.5066, 8.4934), \text{ so we want } P(X = 7) + P(X = 8) = .3483 + .3483 = .6966 \]

68.
a. \( h(x; 6,4,11) \)

b. \( 6 \cdot \left( \frac{4}{11} \right) = 2.18 \)
69.  
   a.  \( h(x; 10,10,20) \) (the successes here are the top 10 pairs, and a sample of 10 pairs is drawn from among the 20)

   b.  Let \( X \) = the number among the top 5 who play E-W. Then \( P(\text{all of top 5 play the same direction}) = P(X = 5) + P(X = 0) = h(5;10,5,20) + h(5;10,5,20) \)

\[
= \binom{15}{5} \binom{15}{10} + \binom{20}{10} \binom{20}{10} = .033
\]

   c.  \( N = 2n; M = n; n = n \)

\[ h(x;n,n,2n) \]

\[ E(X) = n \cdot \frac{n}{2n} = \frac{1}{2} n; \]

\[ V(X) = \left( \frac{2n-n}{2n-1} \right) \cdot n \cdot \frac{n}{2n} \cdot \left( 1 - \frac{n}{2n} \right) = \left( \frac{n}{2n-1} \right) \cdot \frac{n}{2} \cdot \left( 1 - \frac{n}{2n} \right) \cdot \left( \frac{n}{2n-1} \right) \cdot \frac{n}{2} \cdot \frac{1}{2} \]

70.  
   a.  \( h(x;10,15,50) \)

   b.  When \( N \) is large relative to \( n \), \( h(x;n,M,N) \approx b \left( x; n, \frac{M}{N} \right) \)

   so \( h(x;10,150,500) \approx b(x;10,.3) \)

   c.  Using the hypergeometric model, \( E(X) = 10 \cdot \left( \frac{150}{500} \right) = 3 \) and

\[
V(X) = \frac{490}{499} (10)(.3)(.7) = .982(2.1) = 2.06
\]

Using the binomial model, \( E(X) = 10(.3) = 3 \), and

\[
V(X) = 10(.3)(.7) = 2.1
\]
Chapter 3: Discrete Random Variables and Probability Distributions

71.  
   a.  With S = a female child and F = a male child, let X = the number of F’s before the 2\textsuperscript{nd} S.  Then P(X = x) = nb(x;2,.5)

   b.  P(exactly 4 children) = P(exactly 2 males)  
        = nb(2;2,.5) = (3)(.0625) = .188

   c.  P(at most 4 children) = P(X \leq 2)  
        = \sum_{x=0}^{2} nb(x;2,.5) = .25 + 2(.25)(.5) + 3(.0625) = .688

   d.  E(X) = \frac{(2)(.5)}{.5} = 2, so the expected number of children = E(X + 2)  
        = E(X) + 2 = 4

72.  
   The only possible values of X are 3, 4, and 5.  
   p(3) = P(X = 3) = P(first 3 are B’s or first 3 are G’s) = 2(.5)^3 = .250  
   p(4) = P(two among the 1\textsuperscript{st} three are B’s and the 4th is a B) + P(two among the 1\textsuperscript{st} three are G’s and the 4th is a G) = 2 \cdot \binom{3}{2}(.5)^4 = .375  
   p(5) = 1 – p(3) – p(4) = .375

73.  
   This is identical to an experiment in which a single family has children until exactly 6 females have been born (since p = .5 for each of the three families), so p(x) = nb(x;6,.5) and E(X) = 6  
   ( = 2+2+2, the sum of the expected number of males born to each one.)

74.  
   The interpretation of “roll” here is a pair of tosses of a single player’s die (two tosses by A or two by B).  With S = doubles on a particular roll, p = \frac{1}{6}.  Furthermore, A and B are really identical (each die is fair), so we can equivalently imagine A rolling until 10 doubles appear.  
   The P(x rolls) = P(9 doubles among the first x – 1 rolls and a double on the x\textsuperscript{th} roll =  
   \left( \frac{x-1}{9} \right) \left( \frac{5}{6} \right)^{x-10} \left( \frac{1}{6} \right)^9 \left( \frac{1}{6} \right) = \left( \frac{x-1}{9} \right) \left( \frac{5}{6} \right)^{x-10} \left( \frac{1}{6} \right)^{10}

   E(X) = \frac{r(1 – p)}{p} = \frac{10(\frac{5}{6})}{\frac{1}{6}} = 10(5) = 50  
   \sigma(X) = \frac{r(1 – p)}{p^2} = \frac{10(\frac{5}{6})}{(\frac{1}{6})^2} = 10(5)(6) = 300
Section 3.6

75.
   a. \( P(X \leq 8) = F(8;5) = .932 \)
   b. \( P(X = 8) = F(8;5) - F(7;5) = .065 \)
   c. \( P(X \geq 9) = 1 - P(X \leq 8) = .068 \)
   d. \( P(5 \leq X \leq 8) = F(8;5) - F(4;5) = .492 \)
   e. \( P(5 < X < 8) = F(7;5) - F(5;5) = .867 - .616 = .251 \)

76.
   a. \( P(X \leq 5) = F(5;8) = .191 \)
   b. \( P(6 \leq X < 9) = F(9;8) - F(5;8) = .526 \)
   c. \( P(X \geq 10) = 1 - P(X < 9) = .283 \)
   d. \( E(X) = \lambda = 10, \quad \sigma_X = \sqrt{\lambda} = 2.83 \), so \( P(X > 12.83) = P(X \geq 13) = 1 - P(X \leq 12) = 1 - .936 = .064 \)

77.
   a. \( P(X \leq 10) = F(10;20) = .011 \)
   b. \( P(X > 20) = 1 - F(20;20) = 1 - .559 = .441 \)
   c. \( P(10 \leq X \leq 20) = F(20;20) - F(9;20) = .559 - .005 = .554 \)
      \( P(10 < X < 20) = F(19;20) - F(10;20) = .470 - .011 = .459 \)
   d. \( E(X) = \lambda = 20, \quad \sigma_X = \sqrt{\lambda} = 4.472 \)
      \( P(\mu - 2\sigma < X < \mu + 2\sigma) = P(20 - 8.944 < X < 20 + 8.944) \)
      \( = P(11.056 < X < 28.944) \)
      \( = P(X \leq 28) - P(X \leq 11) \)
      \( = F(28;20) - F(12;20) ] \)
      \( = .966 - .021 = .945 \)

78.
   a. \( P(X = 1) = F(1;2) - F(0;2) = .982 - .819 = .163 \)
   b. \( P(X \geq 2) = 1 - P(X \leq 1) = 1 - F(1;2) = 1 - .982 = .018 \)
   c. \( P(1^{st} \text{ doesn’t } \cap 2^{nd} \text{ doesn’t}) = P(1^{st} \text{ doesn’t}) \cdot P(2^{nd} \text{ doesn’t}) \)
      \( = (.819)(.819) = .671 \)
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79. \( p = \frac{1}{200}; n = 1000; \lambda = np = 5 \)
   a. \( P(5 \leq X \leq 8) = F(8;5) - F(4;5) = .492 \)
   b. \( P(X \geq 8) = 1 - P(X \leq 7) = 1 - .867 = .133 \)

80.
   a. The experiment is binomial with \( n = 10,000 \) and \( p = .001 \),
      so \( \mu = np = 10 \) and \( \sigma = \sqrt{npq} = 3.161 \).
   b. \( X \) has approximately a Poisson distribution with \( \lambda = 10 \),
      so \( P(X > 10) \approx 1 - F(10;10) = 1 - .583 = .417 \)
   c. \( P(X = 0) \approx 0 \)

81.
   a. \( \lambda = 8 \) when \( t = 1 \), so \( P(X = 6) = F(6;8) - F(5;8) = .313 - .191 = .122 \),
      \( P(X \geq 6) = 1 - F(5;8) = .809 \), and \( P(X \geq 10) = 1 - F(9;8) = .283 \)
   b. \( t = 90 \) min = 1.5 hours, so \( \lambda = 12 \); thus the expected number of arrivals is 12 and the SD
      \( = \sqrt{12} = 3.464 \)
   c. \( t = 2.5 \) hours implies that \( \lambda = 20 \); in this case, \( P(X \geq 20) = 1 - F(19;20) = .530 \) and \( P(X \leq 10) = F(10;20) = .011 \).

82.
   a. \( P(X = 4) = F(4;5) - F(3;5) = .440 - .265 = .175 \)
   b. \( P(X \geq 4) = 1 - P(X \leq 3) = 1 - .265 = .735 \)
   c. Arrivals occur at the rate of 5 per hour, so for a 45 minute period the rate is \( \lambda = (5)(.75) = 3.75 \),
      which is also the expected number of arrivals in a 45 minute period.

83.
   a. For a two hour period the parameter of the distribution is \( \lambda t = (4)(2) = 8 \),
      so \( P(X = 10) = F(10;8) - F(9;8) = .099 \).
   b. For a 30 minute period, \( \lambda t = (4)(.5) = 2 \), so \( P(X = 0) = F(0;2) = .135 \)
   c. \( E(X) = \lambda t = 2 \)
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84. Let $X$ = the number of diodes on a board that fail.
   
   a. $E(X) = np = (200)(.01) = 2$, $V(X) = npq = (200)(.01)(.99) = 1.98$, $\sigma_X = 1.407$
   
   b. $X$ has approximately a Poisson distribution with $\lambda = np = 2$,
      so $P(X \geq 4) = 1 - P(X \leq 3) = 1 - F(3;2) = 1 - .857 = .143$
   
   c. $P($board works properly$) = P($all diodes work$) = P(X = 0) = F(0;2) = .135$
      Let $Y$ = the number among the five boards that work, a binomial r.v. with $n = 5$ and $p = .135$.
      Then
      
      $$P(Y \geq 4) = P(Y = 4 ) + P(Y = 5) = \binom{5}{4}(.135)^4 (.865) + \binom{5}{5}(.135)^5 (.865)^0 = .00144 + .00004 = .00148$$

85. $\alpha = 1/($mean time between occurrences$) = \frac{1}{2} = 2$
   
   a. $\alpha t = (2)(2) = 4$
   
   b. $P(X > 5 ) = 1 - P(X \leq 5) = 1 - .785 = .215$
   
   c. Solve for $t$, given $\alpha = 2$:
      
      $$.1 = e^{-\alpha t}$$
      
      $\ln(.1) = -\alpha t$
      
      $t = \frac{2.3026}{2} = 1.15$ years

86. $E(X) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} = \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} = \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^y}{y!} = \lambda$

87. a. For a one-quarter acre plot, the parameter is $(80)(.25) = 20$,
      so $P(X \leq 16) = F(16;20) = .221$
   
   b. The expected number of trees is $\lambda$(area) = $80(85,000) = 6,800,000$.
   
   c. The area of the circle is $\pi r^2 = .031416$ sq. miles or 20.106 acres. Thus $X$ has a Poisson distribution with parameter 20.106
88. 

a. \[ P(X = 10 \text{ and no violations}) = P(\text{no violations} \mid X = 10) \cdot P(X = 10) \]
\[ = (.5)^{10} \cdot [F(10;10) - F(9;10)] \]
\[ = (.000977)(.125) = .000122 \]

b. \[ P(y \text{ arrive and exactly 10 have no violations}) \]
\[ = P(\text{exactly 10 have no violations} \mid y \text{ arrive}) \cdot P(y \text{ arrive}) \]
\[ = P(10 \text{ successes in } y \text{ trials when } p = .5) \]
\[ = \left( \frac{y}{10} \right) (.5)^{10} (.5)^{y-10} e^{-10} \frac{(10)^y}{y!} = \frac{e^{-10} (5)^y}{10!(y-10)!} \]

\[ = e^{-5} \cdot \frac{5^{10}}{10!} = p(10;5). \]

In fact, generalizing this argument shows that the number of “no-violation” arrivals within the hour has a Poisson distribution with parameter 5; the 5 results from \( \lambda p = 10(.5) \).

89. 

a. No events in \((0, t+\Delta t)\) if and only if no events in \((0, t)\) and no events in \((t, t+\Delta t)\). Thus, \( P_0(t+\Delta t) = P_0(t) \cdot P(\text{no events in } (t, t+\Delta t)) \)
\[ = P_0(0)[1 - \lambda \cdot \Delta t - o(\Delta t)] \]

b. \[ \frac{P_i(t+\Delta t) - P_i(t)}{\Delta t} = -\lambda P_i(t) \frac{\Delta i}{\Delta t} - P_i(t) \cdot o(\Delta t) \]

\[ = -\lambda P_i(t) \frac{\Delta i}{\Delta t} - \lambda P_{i-1}(t) \]

\[ = -\lambda e^{-\lambda t} - \lambda P_0(t) \] as desired.

c. \[ \frac{d}{dt} [e^{-\lambda t}] = -\lambda e^{-\lambda t} = -\lambda P_0(t) \]

\[ = -\lambda e^{-\lambda t} \]

\[ = -\lambda e^{-\lambda t} + k \lambda e^{-\lambda t} \]

\[ = -\lambda e^{-\lambda t} + \lambda P_{k-1}(t) \]

\[ = -\lambda e^{-\lambda t} + \lambda P_{k-1}(t) \] as desired.
Supplementary Exercises

90. Outcomes are (1,2,3)(1,2,4) (1,2,5) … (5,6,7); there are 35 such outcomes. Each having probability \( \frac{1}{35} \). The W values for these outcomes are 6 (=1+2+3), 7, 8, …, 18. Since there is just one outcome with W value 6, \( p(6) = P(W = 6) = \frac{1}{35} \). Similarly, there are three outcomes with W value 9 [(1,2,6) (1,3,5) and 2,3,4], so \( p(9) = \frac{3}{35} \). Continuing in this manner yields the following distribution:

<table>
<thead>
<tr>
<th>W</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(W)</td>
<td>( \frac{1}{35} )</td>
<td>( \frac{1}{35} )</td>
<td>( \frac{2}{35} )</td>
<td>( \frac{3}{35} )</td>
<td>( \frac{4}{35} )</td>
<td>( \frac{5}{35} )</td>
<td>( \frac{6}{35} )</td>
<td>( \frac{7}{35} )</td>
<td>( \frac{8}{35} )</td>
<td>( \frac{9}{35} )</td>
<td>( \frac{10}{35} )</td>
<td>( \frac{11}{35} )</td>
<td>( \frac{12}{35} )</td>
</tr>
</tbody>
</table>

Since the distribution is symmetric about 12, \( \mu = 12 \), and

\[
\sigma^2 = \sum_{w=6}^{18} (w-12)^2 p(w) = \frac{1}{35} [(6)^2(1) + (5)^2(1) + \ldots + (5)^2(1) + (6)^2(1)] = 8
\]

91.

a. \( p(1) = P(\text{exactly one suit}) = P(\text{all spades}) + P(\text{all hearts}) + P(\text{all diamonds}) + P(\text{all clubs}) = 4 \cdot \frac{13}{52} \cdot \frac{4}{5} = .00198 \)

\[
p(2) = P(\text{all hearts and spades with at least one of each}) + \ldots + P(\text{all diamonds and clubs with at least one of each})
= 6 \left[ P(1 \text{ h and } 4 \text{ s}) + P(2 \text{ h and } 3 \text{ s}) + P(3 \text{ h and } 2 \text{ s}) + P(4 \text{ h and } 1 \text{ s}) \right]
= 6 \left[ \frac{13}{52} \cdot \frac{13}{5} + \frac{13}{52} \cdot \frac{12}{5} + \frac{13}{52} \cdot \frac{11}{5} + \frac{13}{52} \cdot \frac{10}{5} \right] = .14592
\]

\[
p(4) = 4P(2 \text{ spades, 1 h, 1 d, 1 c}) = \frac{4 \cdot \binom{13}{2} \cdot \binom{13}{1} \cdot \binom{13}{1} \cdot \binom{13}{1}}{\binom{52}{5}} = .26375
\]

\[
p(3) = 1 - [p(1) + p(2) + p(4)] = .58835
\]

b. \[
\mu = \sum_{x=1}^{4} x \cdot p(x) = 3.114, \sigma^2 = \sum_{x=1}^{4} x^2 \cdot p(x) - (3.114)^2 = .405, \sigma = .636
\]
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92. \[ p(y) = P(Y = y) = P(y \text{ trials to achieve } r \text{ S's}) = P(y-r \text{ F's before } r^{th} \text{ S}) \]
\[ = nb(y - r, r, p) = \left(\begin{array}{c} y - 1 \\ r - 1 \end{array}\right) p^r (1 - p)^{y-r}, \quad y = r, r+1, r+2, \ldots \]

93. 
   a. \( b(x; 15, .75) \)
   b. \( P(X > 10) = 1 - B(9; 15, .75) = 1 - .148 \)
   c. \( B(10; 15, .75) - B(5; 15, .75) = .314 - .001 = .313 \)
   d. \( \mu = (15)(.75) = 11.75, \quad \sigma^2 = (15)(.75)(.25) = 2.81 \)
   e. Requests can all be met if and only if \( X \leq 10 \), and \( 15 - X \leq 8 \), i.e. if \( 7 \leq X \leq 10 \), so \( P(\text{all requests met}) = B(10; 15, .75) - B(6; 15, .75) = .310 \)

94. \( P(6-v \text{ light works}) = P(\text{at least one 6-v battery works}) = 1 - P(\text{neither works}) \)
\( = 1 - (1 - p)^2. \)
\( P(D \text{ light works}) = P(\text{at least 2 d batteries work}) = 1 - P(\text{at most 1 D battery works}) = 1 - [(1 - p)^2 + 4(1 - p)^3]. \)
\( \text{The 6-v should be taken if } 1 - (1 - p)^2 \geq 1 - [(1 - p)^2 + 4(1 - p)^3]. \)
\( \text{Simplifying, } 1 \leq (1 - p)^2 + 4p(1 - p) \Rightarrow 0 \leq 2p - 3p^3 \Rightarrow p \leq \frac{2}{3}. \)

95. Let \( X \sim Bin(5, .9). \) Then \( P(X \geq 3) = 1 - P(X \leq 2) = 1 - B(2; 5, .9) = .991 \)

96. 
   a. \( P(X \geq 5) = 1 - B(4; 25, .05) = .007 \)
   b. \( P(X \geq 5) = 1 - B(4; 25, .10) = .098 \)
   c. \( P(X \geq 5) = 1 - B(4; 25, .20) = .579 \)
   d. All would decrease, which is bad if the % defective is large and good if the % is small.

97. 
   a. \( N = 500, \quad p = .005, \) so \( np = 2.5 \) and \( b(x; 500, .005) = \mathcal{P}(x; 2.5), \) a Poisson p.m.f.
   b. \( P(X = 5) = p(5; 2.5) - p(4; 2.5) = .9580 - .8912 = .0668 \)
   c. \( P(X \geq 5) = 1 - p(4; 2.5) = 1 - .8912 = .1088 \)

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98. \( X \sim B(x; 25, p) \).
   a. \( B(18; 25, .5) - B(6; 25, .5) = .986 \)
   b. \( B(18; 25, .8) - B(6; 25, .8) = .220 \)
   c. With \( p = .5 \), \( P(\text{rejecting the claim}) = P(X \leq 7) + P(X \geq 18) = .022 + [1 - .978] = .022 + .022 = .044 \)
   d. The claim will not be rejected when \( 8 \leq X \leq 17 \).
      With \( p = .6 \), \( P(8 \leq X \leq 17) = B(17; 25, .6) - B(7; 25, .6) = .846 - .001 = .845 \).
      With \( p = .8 \), \( P(8 \leq X \leq 17) = B(17; 25, .8) - B(7; 25, .8) = .109 - .000 = .109 \).
   e. We want \( P(\text{rejecting the claim}) = .01 \). Using the decision rule “reject if \( X = 6 \) or \( X \geq 19 \)” gives the probability .014, which is too large. We should use “reject if \( X = 5 \) or \( X \geq 20 \)” which yields \( P(\text{rejecting the claim}) = .002 + .002 = .004 \).

99. Let \( Y \) denote the number of tests carried out. For \( n = 3 \), possible \( Y \) values are 1 and 4. \( P(Y = 1) = P(\text{no one has the disease}) = (.9)^3 = .729 \) and \( P(Y = 4) = .271 \), so \( E(Y) = (1)(.729) + (4)(.271) = 1.813 \), as contrasted with the 3 tests necessary without group testing.

100. Regard any particular symbol being received as constituting a trial. Then \( p = P(S) = P(\text{symbol is sent correctly or is sent incorrectly and subsequently corrected}) = 1 - p_1 + p_1 p_2 \). The block of \( n \) symbols gives a binomial experiment with \( n \) trials and \( p = 1 - p_1 + p_1 p_2 \).

101. \( p(2) = P(X = 2) = P(S \text{ on #1 and S on #2}) = p^2 \)
   \( p(3) = P(S \text{ on #3 and S on #2 and F on #1}) = (1 - p)p^2 \)
   \( p(4) = P(S \text{ on #4 and S on #3 and F on #2}) = (1 - p)p^2 \)
   \( p(5) = P(S \text{ on #5 and S on #4 and F on #3 and no 2 consecutive S’s on trials prior to #3}) = [1 - p(2)](1 - p)p^2 \)
   \( p(6) = P(S \text{ on #6 and S on #5 and F on #4 and no 2 consecutive S’s on trials prior to #4}) = [1 - p(2) - p(3)](1 - p)p^2 \)
   In general, for \( x = 5, 6, 7, \ldots \): \( p(x) = [1 - p(2) - \ldots - p(x - 3)](1 - p)p^2 \)
   For \( p = .9 \),
   \[
   \begin{array}{c|cccccccc}
   x & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
   \hline
   p(x) & .81 & .081 & .081 & .0154 & .0088 & .0023 & .0010 \\
   \end{array}
   \]
   So \( P(X \leq 8) = p(2) + \ldots + p(8) = .9995 \)

102. \( a. \) With \( X \sim Bin(25, .1) \), \( P(2 \leq X \leq 6) = B(6; 25, .1) - B(1; 25, .1) = .991 - .271 = 720 \)
   \( b. \) \( E(X) = np = 25(.1) = 2.5, \sigma_X = \sqrt{npq} = \sqrt{25(.1)(.9)} = \sqrt{2.25} = 1.50 \)
   \( c. \) \( P(X \geq 7 \text{ when } p = .1) = 1 - B(6; 25, .1) = 1 - .991 = .009 \)
   \( d. \) \( P(X \leq 6 \text{ when } p = .2) = B(6; 25, .2) = .780 \), which is quite large
103. a. Let event $C = \text{seed carries single spikelets}$, and event $P = \text{seed produces ears with single spikelets}$. Then $P(P \cap C) = P(P | C) \cdot P(C) = .29 \cdot (.40) = .116$. Let $X =$ the number of seeds out of the 10 selected that meet the condition $P \cap C$. Then $X \sim \text{Bin}(10, .116)$. 

$$P(X = 5) = \binom{10}{5} (.116)^5 (.884)^5 = .002857.$$ 

b. For 1 seed, the event of interest is $P = \text{seed produces ears with single spikelets}$. 

$$P(P) = P(P \cap C) + P(P \cap C') = .116 \text{ (from a)} + P(P | C') \cdot P(C') = .116 + (.26)(.40) = .272.$$ 

Let $Y =$ the number out of the 10 seeds that meet condition $P$. Then $Y \sim \text{Bin}(10, .272)$, and $P(Y = 5) = .0767$. 

$$P(Y \leq 5) = \binom{10}{0} (.272)^0 (.728)^1 + \cdots + \binom{10}{5} (.272)^5 (.728)^5 = .97024.$$ 

104. With $S = \text{favored acquittal}$, the population size is $N = 12$, the number of population $S$’s is $M = 4$, the sample size is $n = 4$, and the p.m.f. of the number of interviewed jurors who favor acquittal is the hypergeometric p.m.f. $h(x; 4, 4, 12)$. $E(X) = 4 \cdot \frac{4}{12} = 1.33$.

105. a. $P(X = 0) = F(0; 2) 0.135$ 

b. Let $S =$ an operator who receives no requests. Then $p = .135$ and we wish $P(4 \text{ S’s in 5 trials}) = \binom{5}{4} (.135)^4 (.884)^1 = .00144$ 

c. $P(\text{all receive x}) = P(\text{first receives x}) \cdot \cdots \cdot P(\text{fifth receives x}) = \frac{e^{-2} 2^x}{x!},$ and $P(\text{all receive the same number})$ is the sum from $x = 0$ to $\infty$.

106. $P(\text{at least one}) = 1 - P(\text{none}) = 1 - e^{-\lambda R^2} \cdot \left(\frac{\lambda^2 \pi R^2}{0!}\right)^0 = 1 - e^{-\lambda \pi R^2} = .99 \Rightarrow e^{-\lambda \pi R^2} = .01$ 

$$\Rightarrow R^2 = -\frac{\ln(.01)}{\lambda \pi} = .7329 \Rightarrow R = .8561.$$ 

107. The number sold is $\text{min} (X, 5)$, so $E[\text{min}(X, 5)] = \sum_{x=0}^{5} \min(x, 5) p(x; 4)$ 

$$= (0)p(0;4) + (1)p(1;4) + (2)p(2;4) + (3)p(3;4) + (4)p(4;4) + 5 \sum_{x=5}^{\infty} p(x;4)$$ 

$$= 1.735 + 5[1 - F(4;4)] = 3.59$$
108.  

a.  
\[ P(X = x) = P(A\ wins\ in\ x\ games) + P(B\ wins\ in\ x\ games) \]
\[ = P(9\ S's\ in\ 1^{st}\ x-1\ \cap\ S\ on\ the\ x^{th}) + P(9\ F's\ in\ 1^{st}\ x-1\ \cap\ F\ on\ the\ x^{th}) \]
\[ = \binom{x-1}{9} p^9 (1 - p)^{x-10} p + \binom{x-1}{9} (1 - p)^9 p^{x-10} (1 - p) \]
\[ = \binom{x-1}{9} p^{10} (1 - p)^{x-10} + (1 - p)^{10} p^{x-10} \]

b.  
Possible values of X are now 10, 11, 12, … (all positive integers \( \geq 10 \)). Now
\[ P(X = x) = \binom{x-1}{9} p^{10} (1 - p)^{x-10} + q^{10} (1 - q)^{x-10} \]
for \( x = 10, \ldots , 19 \),

So \( P(X \geq 20) = 1 - P(X < 20) \) and \( P(X < 20) = \sum_{x=10}^{19} P(X = x) \)

109.  

a.  
No; probability of success is not the same for all tests

b.  
There are four ways exactly three could have positive results. Let D represent those with the disease and D' represent those without the disease.

<table>
<thead>
<tr>
<th>Combination</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>D 0 D' 3</td>
<td>[ \binom{5}{0} (.2)^0 (.8)^5 \cdot \binom{5}{3} (.9)^3 (.1)^2 ] [ = (.32768)(.0729) = .02389 ]</td>
</tr>
<tr>
<td>1 2</td>
<td>[ \binom{5}{1} (.2)^1 (.8)^4 \cdot \binom{5}{2} (.9)^2 (.1)^3 ] [ = (.4096)(.0081) = .00332 ]</td>
</tr>
<tr>
<td>2 1</td>
<td>[ \binom{5}{2} (.2)^2 (.8)^3 \cdot \binom{5}{1} (.9)^1 (.1)^4 ] [ = (.2048)(.00045) = .0009216 ]</td>
</tr>
</tbody>
</table>
| 3 0        | \[ \binom{5}{3} (.2)^3 (.8)^2 \cdot \binom{5}{0} (.9)^0 (.1)^5 \] \[ = (.0512)(.00001) = .000000512 \] 

Adding up the probabilities associated with the four combinations yields 0.0273.
Chapter 3: Discrete Random Variables and Probability Distributions

110. \( k(r,x) = \frac{(x + r - 1)(x + r - 2)...(x + r - x)}{x!} \)

With \( r = 2.5 \) and \( p = .3 \), \( p(4) = \frac{(5.5)(4.5)(3.5)(2.5)}{4!} \cdot (.3)^2 \cdot (.7)^4 = .1068 \)

Using \( k(r,0) = 1 \), \( P(X \geq 1) = 1 - p(0) = 1 - (.3)^{2.5} = .9507 \)

111.

a. \( p(x; \lambda, \mu) = \frac{1}{x!} (\lambda^x \mu^{x-1}) e^{-\lambda \mu} \) where both \( p(x; \lambda) \) and \( p(x; \mu) \) are Poisson p.m.f.'s and thus \( \geq 0 \), so \( p(x; \lambda, \mu) \geq 0 \). Further,

\[
\sum_{x=0}^{\infty} p(x; \lambda, \mu) = \frac{1}{2} \sum_{x=0}^{\infty} p(x; \lambda) + \frac{1}{2} \sum_{x=0}^{\infty} p(x; \mu) = \frac{1}{2} + \frac{1}{2} = 1
\]

b. \(.6 p(x; \lambda) + .4 p(x; \mu)\)

c. \( E(X) = \sum_{x=0}^{\infty} x \left[ \frac{1}{2} p(x; \lambda) + \frac{1}{2} p(x; \mu) \right] = \frac{1}{2} \sum_{x=0}^{\infty} xp(x; \lambda) + \frac{1}{2} \sum_{x=0}^{\infty} xp(x; \mu) = \frac{1}{2} \lambda + \frac{1}{2} \mu = \frac{\lambda + \mu}{2} \)

d. \( E(X^2) = \frac{1}{2} \sum_{x=0}^{\infty} x^2 p(x; \lambda) + \frac{1}{2} \sum_{x=0}^{\infty} x^2 p(x; \mu) = \frac{1}{2} (\lambda^2 + \lambda) + \frac{1}{2} (\mu^2 + \mu) \) (since for a Poisson r.v., \( E(X^2) = V(X) + [E(X)]^2 = \lambda + \lambda^2 \)),

so \( V(X) = \frac{1}{2} [\lambda^2 + \lambda + \mu^2 + \mu] - \left[ \frac{\lambda + \mu}{2} \right]^2 = \left( \frac{\lambda - \mu}{2} \right)^2 + \frac{\lambda + \mu}{2} \)

112.

a. \( b(x + 1; n, p) = \frac{(n - x) \cdot \frac{p}{(x + 1) \cdot (1 - p)}}{b(x; n, p)} > 1 \) if \( np - (1 - p) > x \), from which the stated conclusion follows.

b. \( \frac{p(x + 1; \lambda)}{p(x; \lambda)} = \frac{\lambda}{(x + 1)} > 1 \) if \( x < \lambda - 1 \), from which the stated conclusion follows. If \( \lambda \) is an integer, then \( \lambda - 1 \) is a mode, but \( p(\lambda; \lambda) = p(1 - \lambda; \lambda) \) so \( \lambda \) is also a mode. \( [p(x; \lambda)] \) achieves its maximum for both \( x = \lambda - 1 \) and \( x = \lambda \).
113. \( P(X = j) = \sum_{i=1}^{10} P(\text{arm on track } i \cap X = j) = \sum_{i=1}^{10} P(X = j \mid \text{arm on } i) \cdot p_i \) 
where \( p_k = 0 \) if \( k < 0 \) or \( k > 10 \).

114. \( E(X) = \sum_{x=0}^{n} x \cdot \frac{\binom{M}{x} \binom{N - M}{n - x}}{\binom{N}{n}} = \sum_{x=1}^{n} \frac{M!}{(x-1)!(M-x)!} \frac{N-M}{n-x} \binom{N}{n} \)

\[ n \cdot \frac{M}{N} \sum_{x=1}^{n} \frac{(M-1)}{x-1} \frac{N-M}{n-1} \left( \frac{N-1}{n-1} \right) = n \cdot \frac{M}{N} \sum_{y=0}^{n-1} h(y; n-1, M-1, N-1) = n \cdot \frac{M}{N} \]

115. Let \( A = \{ x : |x - \mu| \geq k\sigma \} \). Then \( \sigma^2 = \sum_A (x - \mu)^2 p(x) \geq (k\sigma)^2 \sum_A p(x) \). But \( \sum_A p(x) = P(X \text{ is in } A) = P(|X - \mu| \geq k\sigma) \), so \( \sigma^2 \geq k^2\sigma^2 \cdot P(|X - \mu| \geq k\sigma) \), as desired.

116. 
- a. For \([0,4]\), \( \lambda = \int_0^4 e^{2+6t} dt = 123.44 \), whereas for \([2,6]\), \( \lambda = \int_2^6 e^{2+6t} dt = 409.82 \)
- b. \( \lambda = \int_0^{0.9997} e^{2+6t} dt = 9.9996 = 10 \), so the desired probability is \( F(15, 10) = .951 \).