MATH 232 – Fall 2009 – Test 1 Name: ____________________

Instructions.

- Be sure that your work gives a clear indication of reasoning. Use notation and terminology correctly.
- No mystry numbers: If you use sage, Mathematica, or your calculator, be sure to write down the commands you used.
- **If you get stuck on a problem, always tell me as much as you know.**

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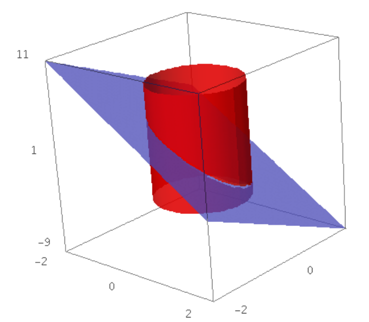
1 Write a parameterization for the line segment from $\langle 1, 2, 3\rangle$ to $\langle -1, 3, 0\rangle$.

2 Write a parameterization for the portion of the plane with equation $3x + 2y + z = 1$ that lies inside the circular cylinder $x^2 + y^2 \leqslant 1$. Be sure to give the ranges of the parameter values.



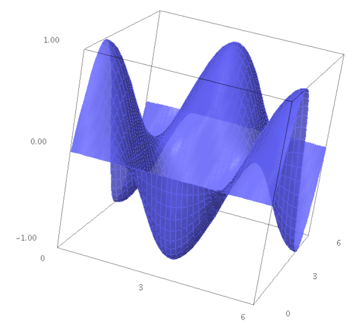
3 Determine the work done by the force field $\boldsymbol{F} = \langle x,\ -z,\ y\rangle$ when moving an object from $\langle 0, 0, 0\rangle$ to $\langle 4, 6, 4\rangle$ along the curve parameterized by $\boldsymbol{r}(t) = \langle 2t, 3t, t^2\rangle$ for $t \in [0, 2]$.

4 Let $\boldsymbol{F} = e^{z}\,\boldsymbol{i} + \boldsymbol{j} + xe^{z}\,\boldsymbol{k}$.

a) Find a potential function for $\boldsymbol{F}$ or explain how we know there is no such potential function.

b) Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where C is the helix parameterized by $\boldsymbol{r}(t) = \langle t, \cos(t\pi), \sin(t\pi) \rangle$ for $t \in [0, 5]$.

5 Set up a double integral with respect to u and v (but **do not evaluate it**) for the surface area of the surface parameterized by $\boldsymbol{r}(u, v) = \langle u,\ v,\ \cos(u)\sin(v)\rangle$ for $u, v \in [0, 2\pi]$. (The shape looks like a portion of an egg carton.) Be sure to include the limits of integration. (If we had a numerical integrator handy, we should be able to get an approximate value from your answer.)



6 Use Stokes' Theorem (if you can, otherwise do something else) to compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F} = \langle y,\ x^2,\ z\rangle$ and C is the path around a triangle from $\langle 1, 0, 0\rangle$ to $\langle 0, 0, 2\rangle$ to $\langle 0, 1, 0\rangle$ and back to $\langle 1, 0, 0\rangle$.

[Note: The triangle lies in the plane with the equation $z = 2 - 2x - 2y$.]

Space for additional work. Please label any work you put here.

Solutions

1 $\boldsymbol{r}(t) = \langle 1,2,3\rangle + \langle -2t, +t, -3t\rangle = \langle 1-2t, 2+t, 3-3t\rangle$

2 We can parameterize using $x = v\cos u$, $y = v\sin u$, and $z = 1-3x-2y$. This gives $\boldsymbol{r}(u,v) = \langle v\cos(u), v\sin(u), 1-3v\cos(u)-2v\sin(u)\rangle$ for $u \in [0,2\pi]$ and $v \in [0,1]$.

3

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^2 \langle 2t, -t^2, 3t\rangle \cdot \langle 2,3,2t\rangle\ dt = \int_0^2 4t - 3t^2 + 6t^2\ dt = 2t^2 + t^3\Big|_0^2 = 8+8 = 16$$

4

a) $\operatorname{curl}\boldsymbol{F} = \langle 0,0,0\rangle$, so $\boldsymbol{F}$ is conservative. The potential function f must satisfy $\frac{\partial f}{\partial x} = e^z$, so $f = xe^z + g(y,z)$. We must also have $\frac{\partial f}{\partial y} = 1$, so $\frac{\partial g}{\partial y} = 1$. This means $g = y + h(z)$, so $f = xe^z + y + h(z)$. Finally, $\frac{\partial f}{\partial z} = xe^z$, so $h'(z) = 0$ and $h(z)$ must be a constant. So $f = xe^z + y + K$ for any constant K is a potential function.

b) We can use the Fundamental Theorem for Line Integrals.

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = f(\boldsymbol{r}(5)) - f(\boldsymbol{r}(0)) = f(5,-1,0) - f(0,1,0) = (5-1) - (0+1)) = 3$$

5 $\boldsymbol{r}_u \times \boldsymbol{r}_v = \langle 1, 0, -\sin(u), v\cos(u), 0\rangle \times \langle 0, \sin(u), 1\rangle =$

$$\text{s.a.} = \int_S 1\ dS = \int_D |\boldsymbol{r}_u \times \boldsymbol{r}_v|\ dA = \int_D |\langle -\sin(u), v\cos(u), 0\rangle \times \langle 0, \sin(u), 1\rangle|\ dA$$

6 An equation for the plane through the three corners is $z = 2-2x-2y$, which we can parameterize as $\boldsymbol{r} = \langle u, v, 2-2u-2v\rangle$ for $x \in [0,1]$ and $0 \leqslant y \leqslant 1-x$.

$\operatorname{curl}\boldsymbol{F} = \langle 0,\ 0,\ 2x-1\rangle$, $\boldsymbol{r}_u = \langle 1,0,-2\rangle$, $\boldsymbol{r}_v = \langle 0,1,-2\rangle$, and $d\boldsymbol{S} = (\boldsymbol{r}_u \times \boldsymbol{r}_v)dA = \langle 2,2,1\rangle dA$.

Putting it all together we get

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_S \operatorname{curl} F \cdot d\boldsymbol{S} = \iint_D \langle 0,0,2u-1\rangle \cdot \langle 2,2,1\rangle dA = \int_0^1 \int_0^{1-u} 2u-1\ dv\ du = -\frac{1}{6}$$

```
sage: integral(integral(2*u-1,v,0,1-u),u,0,1)
-1/6
```

MATH 232 – Fall 2009 – Test 2 Name: ____________________

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- Some R reminders:
  - `require(lattice)` to use lattice graphics.
  - `require(Rcmdr)` to start `Rcmdr`.
  - You may use R's help menus to remind yourself about syntax, etc.

distribution	mass or density function	mean	variance
Poisson	`dpois(x,λ)` $= e^{-\lambda}\dfrac{\lambda^x}{x!}$	λ	λ
Binomial	`dbinom(x,n,π)` $= \binom{n}{x}\pi^x(1-\pi)^{n-x}$	$n\pi$	$n\pi(1-\pi)$
Geometric	`dgeom(x,π)` $= \pi(1-\pi)^x$	$\dfrac{1}{\pi}-1$	$\dfrac{1-\pi}{\pi^2}$
Neg. Binomial	`dnbinom(x,size=s,prob=π)` $= \binom{x+n-1}{s}\pi^n(1-\pi)^x$	$\dfrac{s}{\pi}-s$	$\dfrac{s(1-\pi)}{\pi^2}$
Uniform	`dunif(x,a,b)` $= \dfrac{1}{b-a}$ on $[a,b]$	$\dfrac{b+a}{2}$	$\dfrac{(b-a)^2}{12}$
Std. normal	`dnorm(x,0,1)` $= \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$	0	1
Normal	`dnorm(x,μ,σ)` $= \dfrac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$	μ	σ^2
Lognormal	`dlnorm(x,meanlog=μ,sdlog=σ)` $= \dfrac{1}{\sigma\sqrt{2\pi}}\cdot e^{-[\ln(x)-\mu]^2/2\sigma^2}$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$
Exponential	`dexp(x,λ)` $= \lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
Gamma	`dgamma(x,α,rate=λ)` $= \dfrac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$	α/λ	α/λ^2
Weibull	`dweibull(x,shape=α,scale=β)` $= \dfrac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-(x/\beta)^\alpha}$	$\beta\Gamma(1+\frac{1}{\alpha})$	$\beta^2\left[\Gamma(1+\frac{2}{\alpha})-[\Gamma(1+\frac{1}{\alpha})]^2\right]$
Beta	`dbeta(x,shape1=α,shape2=β)` $= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

1 The mass function for a discrete distribution is described in the table below.

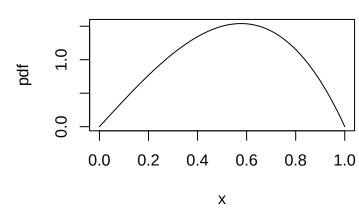
value of Y	0	1	2	3
probability	.5	.25	.15	.1

a) Compute the expected value of this distribution. (Show your work.)

b) Compute the variance of this distribution. (Show your work.)

2 The **pdf** for a continuous distribution is

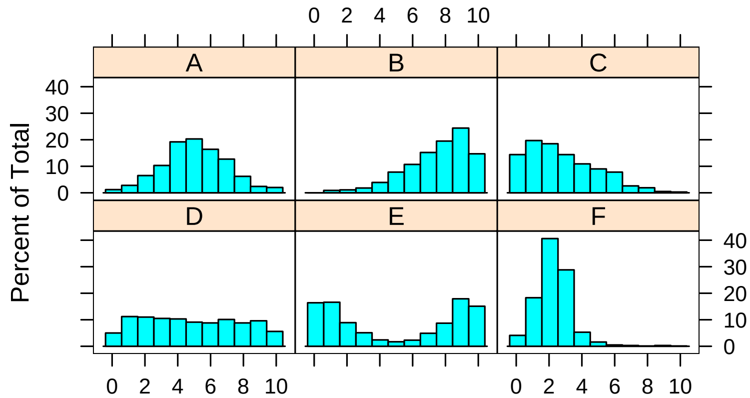
$$f(x) = \begin{cases} 4(x - x^3) & \text{when } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$



a) Determine $P(X \leqslant \frac{1}{2})$.

b) Determine $E(X)$.

3 Histograms of 6 data sets appear below.



a) Which data set has the largest mean?

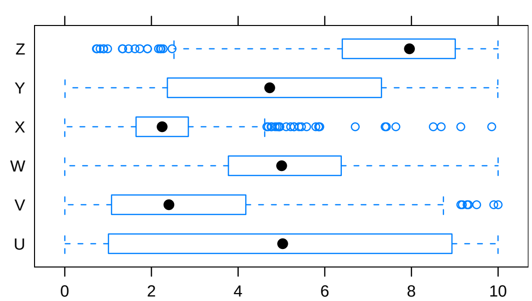
b) Which data set has the largest standard deviation?

c) Which data set has the smallest standard deviation?

d) For which data set or sets is mean clearly smaller than the median? **Explain.**

4 Below are boxplots of the 6 data sets from the previous problem, but with different labels. Match them with their histograms by placing the letters A through F on the blanks.

U: ________ V: ________ W: ________ X: ________ Y: ________ Z: ________



5 A child is playing a game at a local fair. She is told she has a 1/5 probability of winning the game. After playing 5 times and not winning, she is becoming quite angry. She is convinced the game is "not fair", since she has played five times and hasn't won once like she "should have".

a) The child's reasoning reveals that she does not know probability. What distribution can be used to model this situation? Use it to determine the proportion of children who will not win in 5 tries.

b) Convinced by your argument, the child decides to keep playing and has now lost 10 straight times. What are the chances of that?

c) Bonus: Repeat parts (a) and (b) using a different distribution. (You should get the same answers.)

6 Jan is a sales clerk in a small shop. On average, she serves 6 customers per hour. Assume that customers arrive at random times that are independent of one another. Let the X = the number of customers Jan will serve in a 3-hour shift. Let the Y = time until the next customer comes.

a) What distribution is a reasonable model for X? Be sure to specify any parameters of the distribution.

b) What distribution is a reasonable model for Y? Be sure to specify any parameters of the distribution. (You may work in either hours or minutes, but **be sure to say which you are using**.)

c) What is the probability that Jan has 25 or more customers in her 3-hour shift?

d) According to this model, what is the probability that no customers come in the next half hour?

7 If you have ever cooked at high altitude, you know that the decreased atmospheric pressure reduces the boiling point. Now suppose you wanted to estimate the atmospheric pressure on your next high mountain backpacking adventure by measuring the temperature at which water boils. To do this, of course, you need some calibration data showing the boiling temperature and pressure under various conditions.

The following R code loads such a data set and fits a linear model:

```
> boil <- read.csv("http://www.calvin.edu/~rpruim/data/boiling.csv");    # get the data
> summary(lm(Pressure~Temp,boil));

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -64.412751   1.429165  -45.07   <2e-16
Temp          0.440282   0.007444   59.14   <2e-16

Residual standard error: 0.3563 on 29 degrees of freedom
Multiple R-squared: 0.9918,       Adjusted R-squared: 0.9915
F-statistic:  3498 on 1 and 29 DF,  p-value: < 2.2e-16
```

a) Write down an equation for the least squares regression line and use it to predict the pressure if water boils at 190 degrees.

b) Interpret the number 0.9918 in the output above.

c) Why might you be concerned about this linear fit even though r^2 is large? (Use R. Tell me what you checked and what you discovered.)

d) Use a transformation of one or both variables to obtain a better model. Write down your new equation and use it to estimate the pressure when the temperature is 190 degrees. Also tell me why you like this model better.

Solutions

1

a) $E(X) = 1(.25) + 2(.15) + 3(.1) = 0.85$.

b) $E(X^2) = 1(.25) + 4(.15) + 9(.1) = 1.75$, so $V(X) = E(X^2) - [E(X)]^2 = 1.75 - (0.85)^2 = 1.0275$.

2

```
> integrate(function(x) {4*(x-x^3)}, 0,.5)
0.4375 with absolute error < 4.9e-15
> integrate(function(x) {x*4*(x-x^3)}, 0,1)
0.5333333 with absolute error < 5.9e-15
```

3

a) B has the largest mean

b) E has the largest standard deviation

c) F has the smallest standard deviation

d) B has a mean that is smaller than the median becuase the long tail to the left pulls the mean more than the median.

4

Z is the only one skewed left, so it matches B.

X has a smaller variance than V, so X matches F and V matches C.

For the three symmetric distributions, we can go by variance: U mathces E (largest variance), W matches A (least spread), and Y matches D (all quartiles about the same size).

5

```
(4/5)^(c(5,10,15))
[1] 0.32768000 0.10737418 0.03518437
```

6

a) X can be modeled as Poisson with $\lambda = 18$ customers per shift

b) Y can be modeled as Exponential with $\lambda = 6$ customers per hour or $\lambda = 1/10$ customers per minute.

c & d)

```
> 1 - ppois(24,18)        # 25 or more customers when we expect 18 on average
[1] 0.06826021
> 1 - pexp(.5, rate=6)    # first customer comes after at least half hour
[1] 0.04978707
> 1 - pexp(30, rate=1/10) # first customer comes after at least 30 minutes
[1] 0.04978707
> dpois(0,3)              # no customers in half hour, expect 3
[1] 0.04978707
```

7

a) Equation of least squares line: pressure $= -64.41 + 0.440$Temp.

When the temperature is 190 degrees F., we predict the pressure will be $-64.41 + 0.440(190) = 19.24$.

b) This model explains 99.18% of the variation in pressure.

c) If you look a scatter plot you will see a bit of a curve to the pattern in the data. This is even clearer if you look at a residual plot.

d) Several transformations improve the fit.

MATH 232 – Fall 2009 – Test 3 Name: ______________________

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No explanation is required for problems 1 and 2, but you may provide one if you like.

1 If they are computed from the same data, a 95% confidence interval will be

a) wider than

b) narrower than

c) about the same width as

a 90% confidence interval?

2 If they are computed from the same data, a 95% confidence interval will be

a) wider than

b) narrower than

c) about the same width as

a 95% prediction interval?

3 True/False. A campus newspaper reports a 95% confidence interval for the mean weight (in pounds) of male students at that school to be 180 ± 15. This means that 95% of the male students weigh between 165 and 195 pounds. **If you choose false, explain your reasoning.**

4 In the context of statistical inference, what is the difference between a parameter and a statistic? Illustrate the difference with a well-chosen example.

5 A sample of 20 bolts has a mean weight of 15.3 grams with a standard deviation of 0.4 grams. Compute a 95% confidence interval for the mean weight of the bolts.

6 Even when things are running smoothly, 5% of the parts produced by a certain manufacturing process are defective.

a) If you select 10 parts at random, what is the probability that none of them are defective?

Suppose you have a quality control procedure for testing parts to see if they are defective, but that the test procedure sometimes makes mistakes:

- If a part is good, it will fail the quality control test 10% of the time.
- 20% of the defective parts go undetected by the test.

b) What percentage of the parts will fail the quality control test?

c) If a part passes the quality control test, what is the probability that the part is defective?

7 Suppose X an Y are independent random variables with means and standard deviations as given in the following table.

variable	mean	standard deviation
X	40	3
Y	50	4

Determine the mean and standard deviation of the following:

variable	mean	standard deviation
a) $X + Y$		
b) $X - Y$		
c) $\frac{1}{2}X + 7$		

Note: Expected value is another term for mean.

8 An engineering team is testing the interface of a new electronic device. Their plan is to give the device to subjects, give them an instruction, and record whether the first button they push is the correct or incorrect button.

They want to estimate the proportion of people who push the correct button using a 95% confidence interval that has a margin of error no larger than ±5%. How large a sample should the engineers use?

Solutions

1 A. Higher confidence yields wider intervals.

2 B. Prediction intervals are wider.

3 False. The 95% refers to the probability that the confidence interval for a random sample will contain the *mean* of the population. That is, 95% of samples drawn using this method will lead to CI's that contain the true mean of the population.

5 $\overline{x} \pm t_* \frac{s}{\sqrt{n}} = 15.3 \pm 2.093 \frac{0.4}{\sqrt{20}} = 15.3 \pm 0.1872 = (15.11, 15.49) = (15.1, 15.5)$

6

a) $\mathrm{P}(\text{none defective}) = \mathrm{P}(\text{all are good}) = 0.95^{10} = 0.599 = 59.9\%$

b) Even though only 5% are defective, nearly 14% fail the quality control:

$$\begin{aligned} \mathrm{P}(\text{fail test}) &= \mathrm{P}(\text{good and fail}) + \mathrm{P}(\text{bad and fail}) \\ &= \mathrm{P}(\text{good})\,\mathrm{P}(\text{fail} \mid \text{good}) + \mathrm{P}(\text{bad})\,\mathrm{P}(\text{fail} \mid \text{bad}) \\ &= 0.95(0.10) + 0.05(.80) = 0.095 + 0.04 = 0.135 = 13.5\% \end{aligned}$$

c) If a part passes QC, the probability that it is defective drops from 5% to just over 1%. The cost to get this improvement in quality is the cost of the QC test and the 13.5% of manufactured parts that are discarded in the QC process.

$$\begin{aligned} \mathrm{P}(\text{bad} \mid \text{pass}) &= \frac{\mathrm{P}(\text{bad and pass})}{\mathrm{P}(\text{pass})} \\ &= \frac{0.05(0.20)}{0.865} = 0.01156 \end{aligned}$$

7 $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y) = 40 + 50 = 90.$
$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 3^2 + 4^2 = 5^2$. So standard deviation $= 5$.
$\mathrm{E}(X-Y) = \mathrm{E}(X) - \mathrm{E}(Y) = 40 - 50 = -10.$
$\mathrm{Var}(X-Y) = \mathrm{Var}(X + (-Y)) = \mathrm{Var}(X) + \mathrm{Var}(-Y) = 3^2 + 4^2 = 5^2$. So standard deviation $= 5$.
$\mathrm{E}(\frac{1}{2}X + 7) = \frac{1}{2}\mathrm{E}(X) + 7 = 27.$
$\mathrm{Var}(\frac{1}{2}X + 7) = \frac{1}{4}\mathrm{Var}(X) = \frac{9}{4}$. So standard deviation $= \sqrt{\frac{9}{4}}$.

8 margin of error $= z * \sqrt{\pi(1-\pi)/n}$. We don't know π, but we know that the worst case scenario is when $\pi = 0.5$. So we solve for n:

$$\begin{aligned} 0.05 &\geqslant 1.96\sqrt{0.5(0.5)/n} \\ 0.05^2 &\geqslant 1.96^2 0.5^2/n \\ n &\geqslant 1.96^2 0.5^2/(0.05^2) = 384.16 \end{aligned}$$

So a sample of size 385 will do. (If the proportion is not near 0.5, a smaller sample will do.)