Math 162B
Spring, 2002

Test 3 Solutions

1. a) $\lim_{n\to\infty} \frac{n^2}{n\sqrt{3n^2-4}} = \lim_{n\to\infty} \frac{n}{\sqrt{3n^2-4}} = \lim_{n\to\infty} \frac{1}{\sqrt{3-\frac{4}{n^2}}} = \frac{1}{\sqrt{3-0}} = \frac{1}{\sqrt{3}}$

b) $\frac{-1}{n^2} \leq \frac{\sin(n)}{n^2} \leq \frac{1}{n^2}$, and $\lim_{n\to\infty} \frac{\pm 1}{n^2} = 0$, so $\lim_{n\to\infty} \frac{\sin(n)}{n^2} = 0$ by the Squeeze Theorem.

2. $0.1212\overline{12} = \sum_{n=1}^{\infty} 12 \cdot (\frac{1}{100})^n = \frac{\frac{12}{100}}{1-\frac{1}{100}} = \frac{12}{99}$

3. a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is alternating and $\frac{1}{\sqrt{n}}$ decreases with a limit of 0, so by the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges. On the other hand, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} \leq 1$, so the convergence is not absolute.

b) Use the root test: Since $\lim_{n\to\infty} \sqrt[n]{\frac{1}{(2n+3)^n}} = \lim_{n\to\infty} \frac{1}{2n+3} = 0$, $\sum_{n=1}^{\infty} \frac{1}{(2n+3)^n}$ series converges absolutely.

c) If $n > 3$, then $\frac{1}{\sqrt[n]{2n+3}} > \frac{1}{2n+3} > \frac{1}{3n}$.

Since $\sum \frac{1}{3n}$ diverges, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{2n+3}}$ also diverges.

The Term Limit Test can also be used since $\frac{1}{\sqrt[n]{2n+3}} \to 1$ as $n \to \infty$.

4. $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$ converges if and only if $\int_e^{\infty} \frac{dx}{x(\ln(x))^p}$. But $\int_e^{\infty} \frac{dx}{x(\ln(x))^p} = \int_1^{\infty} \frac{du}{u^p}$ (use a substitution $u = \ln(x)$), which converges if and only if $\sum_{n=2}^{\infty} \frac{1}{n^p}$ converges. This is just a p-series, so we have convergence (for all the integrals and all the sums mentioned above) exactly when $p > 1$.

5. a) $\sum_{n=2}^{4} a_n = \frac{1}{3} - \frac{1}{8} + \frac{1}{15}$

b) `Sum[(-1)^n /( (n-1)*(n+1) ), {n,2,400}]`

c) This is an alternating series, so the infinite sum lies between consecutive partial sums, which differ by one term. Using `NSolve[(n-1)(n+1) == 10000, n]` we can find a value of n such that a_n is sufficiently small. High and low estimates are then given by $\sum_{n=2}^{k-1} a_n$ and $\sum_{n=2}^{k} a_n$ where k is an integer at least as large as the solution found by `NSolve[]`.

6. The center of the interval of convergence must be at 2, so only (b) and (f) are possible.

7. a) Use ratio test: $\left|\frac{a_n+1}{a_n}\right| = \frac{|4x|}{n+1} \to 0$ as $n \to \infty$ for all x, so $\sum_{n=0}^{\infty} \frac{(4x)^n}{n!}$ converges for all x.

b) Use ratio test: $\left|\frac{a_n+1}{a_n}\right| = \frac{|x|(n-1)}{n+1} \to |x|$ as $n \to \infty$. So $\sum_{n=2}^{\infty} \frac{x^n}{n(n-1)}$ converges when $|x| < 1$ and diverges when $|x| > 1$. Now we need to check the endpoints. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)}$ converges by the Alternating Series Test. $\sum_{n=2}^{\infty} \frac{(1)^n}{n(n-1)}$ converges because it is pretty much a p-series with $p = 2$. This can be shown more carefully by using the limit comparison test to compare $\frac{(1)^n}{n(n-1)}$ with $\frac{1}{n^2}$.

8. See figure 6 on page 719, but start at 2 rather than at 1.

9. For the statements of how these tests work see pages 733 and 735 or your notes from class. The proof that the ratio test works appears on page 733.

The proof of the root test was a homework assignment (and is easier): Suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$. Pick some value r such that $L < r < 1$. ($r = \frac{L+1}{2}$ works, for example.) For large enough n, we know that $\sqrt[n]{|a_n|} < r$, so $|a_n| < r^n$. Since $\sum r^n$ converges, $\sum |a_n|$, converges too. So $\sum a_n$ converges absolutely.