# Lab D: Sequences and Series

## Tables and Sums

Last time we learned that *Mathematica* has some nice features for making lists of numbers and for adding up the terms in the list. This section serves as a brief reminder of the `Table[]` and `Sum[]` commands. These will be useful tools for working with sequences, partial sums and infinite series.

Suppose, for example, you wanted a list of the first 30 perfect squares. (Using our notations for sequences, we could write this as $\{n^2\}_{n=1}^{30}$.) The following command will give it to you: `Table[n^2, {n,1,30}]`

The list `{n,1,30}` tells *Mathematica* that `n` is the variable to adjust when moving from term to term, and that the lowest value of `n` to use is 1; the highest, 30. By default, *Mathematica* uses a *step size* of 1, that is it tries `n` = 1, 2, 3, 4, ...

You can make a plot of (the beginning of) a sequence using `ListPlot[]`. For example, `ListPlot[Table[n^2,{n,1,30}]]` will plot the first 30 squares.

In addition to making a list (i.e., part of an infinite numerical sequence), *Mathematica* can add up the terms for you. Simply replace `Table` with `Sum` or `NSum[]`. For example, `Sum[n^2, {n,1,30}]` outputs the value of $\sum_{n=1}^{30} n^2$, the sum of the first 30 squares (the result is 9455). Note: It is sometimes handy to first use `Table[]` to make sure that you have set up the sum correctly, since you will be able to see the terms involved listed out. Once you think you have it correct, switch to `Sum[]` to get the answer.

## Some Sums (a.k.a. Series)

**1.** Let $\{a_n\}_{n=1}^{\infty}$ be the sequence where $a_n = \frac{1}{n(n+1)}$.

a) Get *Mathematica* to list the first 20 terms of this sequence.

b) Get *Mathematica* to plot the first 20 terms of this sequence. (You did read how to do this above didn't you?)

c) Get *Mathematica* to list the first 20 partial sums. (Use `Table[]` and `Sum[]` together.)

d) Get *Mathematica* to plot the first 20 partial sums. (Use `ListPlot` and `Table[]` and `Sum[]` together.)

e) You should see a pattern emerging. Based on the pattern, what do you think $\displaystyle\sum_{k=1}^{n} a_k$ is?

f) Use the method of partial fractions to express $\frac{1}{k(k+1)}$ as the sum of two fractions. Use this to prove that the formula you gave for $\displaystyle\sum_{k=1}^{n} a_k$ is indeed correct.[1]

g) Based on this, what is $\displaystyle\sum_{k=1}^{\infty} a_k$? Explain.

**2.** The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series.

a) List and plot the first 40 terms of the sequence $\{a_n\} = \frac{1}{n}$.

b) List and plot the first 40 partial sums.

c) Based on this what do you think $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is?

[1] Do the partial fractions by hand to remind yourself how it goes. You can check your work using `Apart[]` .

d) Use *Mathematica*[2] to determine $\sum_{n=1}^{10^k} \frac{1}{n}$ for $k = 1, 2, 3, \ldots 15$. **Use `NSum[]` rather than `Sum[]` for this or you will be waiting forever for you results.** (Why?) How does that compare with your answer to the previous item?

**3.** Repeat the previous problem for the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for some value of $p$ other than 1. These series are given the (rather dumb) name "$p$-series".

We'll learn more about $p$-series (including the harmonic series) in class next week.

**4.** Investigate $\sum_{n=0}^{\infty} \frac{1}{n!}$. What do you think the value is? We'll learn about this series (and some related series) later.

## Geometric Sequences, Sums, and Series

The most important series are the geometric series. We will encounter them over and over. The problems below give you some more experience with them.

**5.** The Cantor set is a set of real numbers between 0 and 1. It is formed in the following way. Start with $C_0 = [0, 1]$. Let $C_1$ be what is left when you throw away the middle third – $(\frac{1}{3}, \frac{2}{3})$ – from $C_0$. So $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Now throw away the middle third of each of these parts. What is left is $C_2$, etc., etc. The Cantor set $C$ is all points that never get thrown away.

a) Give some examples of points in the Cantor set.

b) $C_0$ has 1 piece. $C_1$ has two pieces. How many pieces are in $C_n$? How long is each piece? What is the total length of all the pieces of $C_n$? What happens as $n \to \infty$?

c) The total amount removed from $C_0$ to get $C_1$ has length $\frac{1}{3}$. The total The total amount removed from $C_1$ to get $C_2$ has length $2 \cdot \frac{1}{3} \cdot \frac{2}{3}$. (Why?) What is the total amount removed from $C_n$ to get $C_{n+1}$?

d) If we add up all the the removed amounts, what do we get? (Hint: geometric series)

e) What would happen if we removed the middle quarter instead of the middle third? What if we removed the middle half?

**6.** Express the repeating decimal $0.0202\overline{02}$ as a geometric series. What is the ratio ($r$)? What is the first term? Use our results about geometric series to express $0.0202\overline{02}$ as an ratio of integers.

**7.** Express $1.234545\overline{45}$ as an ratio of integers.

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[2] Just how far this can be carried out depends on the machine you are using for *Mathematica* and the settings regarding precision. If you get an error message that indicates the *Mathematica*is unable to get the desired precision, or that there is not enough memory for the task you are performing, then try reducing the number 15 until things work. You can stop execution of a command that is running too long by going into the "kernel" menu.