

Getting Started

Do not turn on *Mathematica*. If you already have, turn it off. (It takes too much memory away from other things, and we want to run an applet in Internet Explorer.) Now start up Internet Explorer. From the Mathematics Department home page navigate to the Java Demos for this class and select the Newton's method demonstration. (If it page seems to take a long time to load, try reloading it.)

Newton's Method: What it Does

Remember the idea behind Newton's Method: We want to approximate a value of x such that $f(x) = 0$. Such a points is called a *zero* or a *root* of the function f . As with linearization and differentials, we will use the tangent line approximation to the function. We will use this to get a sequence of approximations $x_0, x_1, x_2, x_3 \dots$, where x_0 is our initial guess.

There are many applications of this. For example, we might like to find critical points (approximately) from a derivative that is too messy to solve algebraically. It can also be used to find numerical approximations to certain constants. Suppose we want to use Newton's method to get an approximation to $\sqrt{2}$. We need to find a function f such that $f(\sqrt{2}) = 0$. Since $(\sqrt{2})^2 = 2$, $(\sqrt{2})^2 - 2 = 0$. So we can use the function $f(x) = x^2 - 2$. Similar things can be done for other constants.

1. Suppose you want to approximate the numbers listed below using Newton's method. Find a function you could use for each.
 - a) $\sqrt{5}$
 - b) $\sqrt[3]{3}$
 - c) π
 - d) e

Newton's Method: The Algebra

Now it is time to work out the algebra involved in Newton's method. Let's suppose that our current estimate for a zero of f is x , that is $f(x) \approx 0$. We want to get a new estimate, which we will call z .

2.
 - a) Draw a large sketch of the graph of a function that has at least one root. Pick a point on the graph of the function (not a root) and label it $(x, f(x))$.
 - b) Draw a portion of the tangent line at the point $(x, f(x))$ that includes where it intersects the horizontal axis.
 - c) How should you label the point where the tangent line intersects the horizontal axis?

The facts we need about the tangent line are:

- The slope of tangent line at a point $x = f'(x)$.
- The tangent line goes through the point $(x, f(x))$.
- Our new estimate for a zero of f will be where the tangent line crosses the horizontal axis.

Let's call the point where the tangent line intersects the horizontal axis $(z, 0)$. (Is that how you labeled it on your graph?) z will be our new estimate. We can determine the value of z by solving an equation that says

$$\text{slope of tangent line} = \text{slope of line through points } (x, f(x)) \text{ and } (z, 0)$$

3.
 - a) What is the slope of the tangent line?
 - b) What is the slope of the line through $(x, f(x))$ and $(z, 0)$.
 - c) Set the two expressions above equal to each other and solve for z

You should have discovered that

$$z = x - \frac{f(x)}{f'(x)}.$$

That is,

$$\text{new estimate} = \text{old estimate} - \frac{f(\text{old estimate})}{f'(\text{old estimate})}.$$

This is called the **Newton's Iteration Formula**. It is usually written with slightly different notation. Remember, we want to repeat this process of getting new estimates from old estimates, so we will get a sequence of estimates $x_0, x_1, x_2, x_3, \dots$. If our current estimate is x_n , we get the next estimate using the iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Regardless of the notation, Newton's iteration formula tells us how to get new estimates from previous estimates.

4. Let $f(x) = x^2 - a$, where a is a constant. Notice that $f(\sqrt{a}) = 0$, so we can use Newton's method to try to approximate \sqrt{a} for any $a > 0$.
 - a) Write out Newton's iteration formula in this case (plugging in the appropriate derivative, etc.) and simplify the expression.
 - b) Now let $a = 5$. Starting with an initial estimate of 1, use your iteration formula to get the next three estimates for $\sqrt{5}$ by hand, expressing your results as fractions. (You can check your work with the applet if you like, but show your work.)
 - c) Compare the third estimate (x_3) with the value your calculator gives for $\sqrt{5}$. How accurate does the estimate seem to be?
 - d) Use the applet to figure out the first 8 estimates. (You will have to enter the appropriate function.) Write them down. Compare the last estimate with the value your calculator gives for $\sqrt{5}$. How accurate does the estimate seem to be?
5. Find an approximation to $\sqrt[3]{3}$ that is accurate to 8 decimal places. Give your initial guess, your final estimate, how many iterations it took, and why you think it is accurate to 8 decimal places. (Note: it is possible to go beyond 8 iterations if needed; do you see how?)

Newton's Method: Behavior

Although Newton's method worked well in the examples above, it is not a solution that works in every situation.

6. Fill in the blank: As we mentioned in class, if the slope of the tangent line is _____, then we cannot get a new estimate from an old estimate. Explain why.
7. Of course, to be able to use Newton's method for estimation, we would like $f(x)$ and $f'(x)$ to be fairly easy to compute (exactly, if possible). Why?

Another problem that can arise is that Newton's method may find a different root than the one we want.

8. Load the example entitled "When else is $\sin(x^2) = \sin(x)^2$?" Set it to show a few iterations and drag the initial estimate around. Notice how sometimes it does not provide a good estimate for the nearest zero. Record the value of one or two of the most "interesting" initial estimates you find.
9. Take a look at the two "problem functions" in the list of examples. Experiment with different initial estimates. Is it possible to find a good starting point for these functions? If so, how must it be chosen? If not, why not?