1. Use *Mathematica*'s help to learn how `Integrate[]` and `NIntegrate[]` work. Use these to compute the following:

a) $\int \sec(x)\, dx$

b) $\int_0^{\pi} \sin(x)\, dx$

c) $\int_0^1 e^{x^2}\, dx$

2. a) Is there a function f such that $f'(x) = x^2 + \sin(x)$ and $f(1) = f(-1)$? If so, find an example. If not, why not?

b) Is there a function g such that $g'(x) = \dfrac{1}{x^2}$ and $g(1) = g(-1)$? If so, find an example. If not, why not?

3. Suppose that $f(x)$ and $g(x)$ have definite integrals with the following properties:

$$\int_0^2 f(x)\,dx = 3 \qquad \int_0^4 f(x)\,dx = 6 \qquad \int_4^8 f(x)\,dx = 1 \qquad \int_0^8 g(x)\,dx = -5$$

Find the following definite integrals:

$$\int_2^0 f(x)\,dx \qquad \int_2^4 f(x)\,dx \qquad \int_0^8 f(x)\,dx \qquad \int_0^8 3f(x) + 2g(x)\,dx$$

4. a) Which of the following is true? If f is integrable on $[a, b]$ then

$$i)\ \left|\int_a^b f(x)\ dx\right| \le \int_a^b |f(x)|\ dx \qquad ii)\ \int_a^b |f(x)|\ dx \le \left|\int_a^b f(x)\ dx\right|$$

b) Prove the one in a) that is true and give a counter-example for the one in a) that is false.

c) Using the above, prove that

$$i)\ \left|\int_0^1 \frac{\cos nx}{x+1}\ dx\right| \le \ln 2 \text{ for all n} \qquad ii)\ \left|\int_0^{e-1} \frac{\sin nx}{x+1}\ dx\right| \le 1 \text{ for all n}$$

5. What is wrong with the following argument?

$$\int_{-1}^2 \frac{dt}{t^2} = \left.\frac{-1}{t}\right|_{-1}^2 = -\frac{1}{2} - \left(-\frac{1}{-1}\right) = -\frac{1}{2} - 1 = -\frac{3}{2}$$

6. a) The region under the graph of $y = -2x + 4$ on $[-2, 1]$ is to be divided into two parts of equal area by a vertical line. Where should the line be drawn?

b) Where would you draw a horizontal line to divide the region in part a) into two parts of equal area?

7. Consider the area enclosed by the curve $y = x^2$, the $x-$axis, the lines $x = 1$ and $x = 2$. What vertical line will divide this area into two equal parts?

8. What horizontal line will divide the area enclosed by the curve $y = \dfrac{1}{x^2}$, the $x-$axis, the lines $x = 1$ and $x = 2$ into two equal parts?

9. Find all continuous functions $f(x)$ satisfying

$$\int_0^x f(t)\ dt = [f(x)]^2 + C$$

(Hint: Differentiate both sides with respect to x.)

10. Let $v(t)$ be the velocity and $s(t)$ be the position of some object at time t.

a) Write down a formula for computing the average velocity over the time interval $[a, b]$.

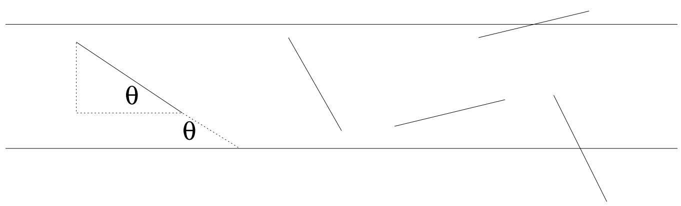
b) Use the Fundamental Theorem of Calculus (Second Part) to rewrite this so it involves an integral.

c) Notice that you now have two (equivalent) expressions for the average velocity over an interval. Based on the second one, how do you think mathematicians define the average value of a function over an interval?

d) How does this definition compare to the average value of 100 test scores? (Hint: think Riemann sums.)

11. TOOTHPICKS ON A HARDWOOD FLOOR.

Imagine a hardwood floor which has wooden panels 5 cm wide arranged parallel along the floor. The grooves between adjacent panels form parallel lines 5 cm apart. If you drop a 5 cm long toothpick on the floor, what is the chance that it will straddle one of these grooves?



a) When the toothpick lands, it must form some angle relative to the parallel grooves. Call this angle θ. For a fixed value of θ, let $P(\theta)$ be the probability that the toothpick straddles a groove.

  i. What is $P(\frac{\pi}{6})$?

  ii. Express $P(\theta)$ as a function of θ.

b) Now determine the average value of $P(\theta)$ (over what interval?). Since each angle is equally likely, this should give you the probability that the toothpick straddles a groove.

c) Now suppose the grooves are spaced at distance of l from each other and the length of the toothpick is T. Now what is the probability that the toothpick straddles a groove?

d) Try it out. Find a suitable object (a toothpick or a pencil) and a floor with parallel lines (tile, hardwood, etc.) Toss your object 10 or 20 times and see how many times it straddles the lines. How close was this to the theoretical value? If the values are not very close, see if you can explain why.