DO NOT start ***Mathematica*****.**
If you already have, please exit ***Mathematica*** **before continuing.**

1 The definition

Recall the definition of limit: $\lim_{x \to a} f(x) = L$ means

for any $\varepsilon > 0$,	no matter what positive number Alice picks (ε)
one can find a number $\delta > 0$,	Bob can find a positive number δ
such that	such that
if $0 < \lvert x - a \rvert < \delta$,	if x is within δ of a (but $x \neq a$)
then $\lvert f(x) - L \rvert < \varepsilon$.	then $f(x)$ is as close to L as Alice specified.

Note: The Greek letters used above are called epsilon (ε) and delta (δ).

One can think of the definition of limits in terms of a game between two players, Alice and Bob. The game works as follows: Bob proposes a value of the limit L, Alice then challenges Bob to find an interval near a such that $f(x)$ is within some small distance (ε) from the limit L, that is Alice chooses ε and challenges Bob to make sure that $f(x)$ should be in the interval $(L - \varepsilon, L + \varepsilon)$. Bob must then specify the interval of x values by providing the number δ, i.e., the interval $(x - \delta, x + \delta)$. Bob wins if every x in the interval $(x - \delta, x + \delta)$ (except possibly $x = a$) satisfies $f(x) \in (L - \varepsilon, L + \varepsilon)$. Alice wins if there is some x in the interval $(x - \delta, x + \delta)$ (but not $x = a$) such that $\lvert f(x) - L \rvert > \varepsilon$. If the limit is indeed L, Bob will be able to find a δ for any such ε chosen by Alice. That is, Bob can always win. If there is some ε that Alice could pick for which Bob has no winning δ, then Alice can win and the limit is not L.
We can picture these intervals on a graph like the one below:

2 Applying the Definition

The Epsilon-Delta Applet provides an interactive version of this picture that we will use in this lab to explore the definition of limit a bit further. Load the applet using Internet Explorer (not Netscape). Select "Epsilon-Delta Applet" from the list at

`http://www.calvin.edu/~rpruim/courses/m161/F01/java/`

Each function below has already been entered in the examples menu of the applet. **Remember:** ***Mathematica*** **should NOT be running.**

1. Let $f(x) = 6x - x^2$. Consider $\lim_{x \to 2} f(x)$.

a) What is $\lim_{x \to 2} f(x)$? How does this show up on the graph?

b) If Alice picks $\varepsilon = 0.3$ and Bob picks $\delta = 0.1$, who wins?

c) If Alice picks $\varepsilon = 0.03$ and Bob picks $\delta = 0.01$, who wins?

d) If Alice picks $\varepsilon = 0.003$ and Bob picks $\delta = 0.001$, who wins?

e) Based on the results above and the graphs you have looked at, if Alice picks some $\varepsilon > 0$, what do you think Bob should pick for δ? (You don't have to prove that this works, but it does as long as ε is small.)

2. Let $f(x) = x^2 - 2x - 1$. Consider $\lim_{x\to 3} f(x)$.

a) Since f is a _______________, $L = \lim_{x\to 1} f(x)$ is easy to compute, namely $L =$ ________. Enter this value on the second variable input line where it says "test limit $L =$". (The value of $a = 3$ has already been set correctly.)

b) If Alice picks $\varepsilon = \frac{1}{2}$ and Bob picks $\delta = \frac{1}{4}$, who wins? Explain how you know in terms of the graph given by the applet. (You may need to zoom in.)

c) If Alice picks $\varepsilon = 1$ and Bob picks $\delta = \frac{1}{5}$, who wins? Explain how you know in terms of the graph given by the applet. (You may need to zoom in.)

d) If Alice picks $\varepsilon = 1$, what is the largest value Bob can pick for δ and still win? (Approximate this as well as you can using the graph.)

3. Let $f(x) = x^2 + 1$. Consider $\lim_{x\to 0} f(x)$.

a) What is $\lim_{x\to 0} f(x)$?

b) If Alice picks $\varepsilon = 0.25$, what should Bob choose for δ?

c) If Alice picks $\varepsilon = 0.09$, what should Bob choose for δ?

d) Show that if Alice picks ε and Bob picks $\delta = \sqrt{\varepsilon}$, then Bob wins. (This proves that the limit is what Bob claims it is.)

4. Let $f(x)$ be the function in Example 4 of the applet. $\lim_{x\to 2} f(x)$ does not exist. This means Alice should always be able to win the game. How should Alice pick ε in order to win?

5. Let $f(x)$ be the function in Example 5 of the applet. Does $\lim_{x\to 1} f(x)$ exist? Explain.

6. Let $f(x) = 2x\sin(1/x)$.

a) What is $\lim_{x\to 0} f(x)$?

b) If Alice picks $\varepsilon = 0.5$, what should Bob choose for δ?

c) If Alice picks $\varepsilon = 0.1$, what should Bob choose for δ?

d) If Alice picks $\varepsilon = 0.01$, what should Bob choose for δ?

e) If Alice picks some $\varepsilon > 0$, what should Bob choose for δ? Prove that your choice works.