Differentiation with *Mathematica*

You may work in groups of 2 or 3 for this lab.

How *Mathematica* Does Derivitives

Mathematica can find derivatives for you. In fact, there are two ways to get *Mathematica* to find a derivative for you.

The Prime Method

If you have defined a function (e.g. `f[x_]:= Sin[x]`), then `f'[x]` will give you the derivative.

1. Use this method to compute $f'(x)$ where $f(x) = x^2 + 3x$. (Be sure to do `Clear[f]` if you have already used `f` this session.) What is $f'(4)$?

The `D[]` method

A second way to compute derivatives is to use the *Mathematica* function `D[]`. The syntax is `D[`*expression*`,`*variable*`]`.

2. Repeat problem 1 using this method.

To get the value of the derivative at the point $x = 4$, you can use the following command:

```
D[x^2+3x, x] /.x->4
```

The `./x->4` says to substitute 4 for x (after finding the derivative).

3. For each of the following, use *Mathematica* to find $\frac{dy}{dx}$ and $\left.\frac{dy}{dx}\right|_{x=2}$

a) $y = (x^2 - 4x + 7)/(2x^2 + 5x + 7)$

b) $y = \cos(1 + x^2)$

c) $y = (2x + 3)^5(3x - 8)^7$

Some Problems to Do

For the remainder of the period, work on the following problems. Use *Mathematica* whenever it is required or helpful. You might like to think about how to get *Mathematica* to help you, even if you know how to do things by hand or with a calculator already. I've included some *Mathematica* hints along the way, too.

Your **report** should be a careful write up of solutions to problems 4–6 and any additional problems from 7–10 that you complete. (You need not report on problems 1–3.) Take notes as you work on the problems and then have different members of your groups write solutions to different problems. Once you have finished problem 6 you may work on the remaining problems in any order you choose. You won't have time to do them all during class, but report on the ones you get done.

4. This one requires *Mathematica*. *Mathematica* knows all the rules you know (and then some) for differentiation. For example, type `D[g[x]*h[x],x]` and see what you get. (Clear g and h first if you have already defined them. Note that the square brackets tell *Mathematica* that g and h are functions of x, even though they have not been defined. What happens in you leave off the `[x]`? Why?)

a) Get *Mathematica* to give you the Quotient Rule and Chain Rule.

b) For the quotient rule, you probably got something that looks a little different from the form we wrote in class. Use `Together[]` to force *Mathematica* to express things as one fraction.

Here is an example of the syntax: `Together[7/x + 10/(x * x)]` will result in $\frac{10+7x}{x^2}$. You can also type

```
7/x + 10/(x * x) // Together
```

`Together[]` is much like `Simplify[]` in that it changes the form of an expression but not its value. Algebraic manipulations *Mathematica* can do include

`Simplify[]`	tries to get a "simple" expression
`Together[]`	puts over common denominator
`Cancel[]`	tries to remove common terms from fractions
`Factor[]`	tries to express things as products of terms
`Expand[]`	perfroms multiplications to write as a sum of terms
`Apart[]`	tries to get simple denominators

5. It is possible to take derivatives of derivatives. For example, if $f(x) = x^3$, then $f'(x) = \frac{df}{dx} = 3x^2$, and the second derivative is $f''(x) = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(3x^2\right) = 6x$. Of course, the third derivative is $f'''(x) = \frac{d^3f}{dx^3} = 6$. And so on.

a) Let $g(x) = \sin(x)$. Determine $g'(x)$, $g''(x)$, $g'''(x)$, and $g''''(x) = g^{(4)}(x)$.

b) What is $g^{(59)}(x)$ (the 59th derivative of g)? How do you know?

6. Find all points on the parabola with the equation $y = (x-4)(x-9)$ such that either the tangent line or the normal line[1] at the point goes through the origin. Hints: (1) Draw a picture (by hand). (2) For a point on the parabola, determine a) the slope of the tangent (or normal) line and b) the slope of the line joining that point to the origin. What does it mean if these slopes are the same? What does it mean if they are different?

7. a) Let $F(x) = g(x)h(x)$. Compute F', F'', F''', and $F''''(x)$ in terms of g, h, and their derivatives. (*Mathematica* hint: `D[f[x],{x,`*n*`}]` will give the *n*th derivative.)

b) Compute $(A+B)^n$ for $n = 1, 2, 3,$ and 4. (*Mathematica* hint: Use `Expand[]` to get *Mathematica* to expand $(A+B)^n$ for you.)

c) Compare the results above. What do you notice?

8. Assume that $f(x)$ is a differentiable function, and that the values of f and its derivatives at the points $x = 0$, 1, 2, and 3 are given as follows:

$f(0) = 3$	$f(1) = 5$	$f(2) = -2$	$f(3) = 6$
$f'(0) = -1$	$f'(1) = 0$	$f'(2) = 3$	$f'(3) = 1$

Let $g(x) = x^2 - 3x + 2$.

Calculate $\frac{d}{dx}\big(f(x)g(x)\big)$ at $x = 0$, 1, 2, and 3.

9. Let f be the function such that $f(x)$ is the value of the sine of an angle which measures x **degrees**. (NOTE: For most values of x, $f(x) \neq \sin x$. Why not?) Let g be defined similarly for cosine.

a) Express f and g in terms of sin and cos.

b) What is $\frac{df}{dx}$? What is $\frac{dg}{dx}$? (HINT: Use part a) and the chain rule.)

c) Express $\frac{df}{dx}$ and $\frac{dg}{dx}$. in terms of $f(x)$ and $g(x)$. (No mention of sin or cos allowed.)

d) Is it still true that $(f(x))^2 + (g(x))^2 = 1$?

e) Why don't we use degrees in calculus?

10. Let $f(x) = x^3 + 3x^2 + 3x + 5$.

a) Does f have an inverse? How do you know? How could a derivative and the Intermediate Value Theorem be useful in figuring this out?

b) If f has an inverse, determine $f^{-1}(5)$, $\left(f^{-1}\right)'(5)$, $f^{-1}(12)$, and $\left(f^{-1}\right)'(12)$. Do this without getting an explicit formula for f^{-1}. (We have not learned how to compute derivatives of inverses yet, but you should be able to figure this out from the geometry. A `Plot[]` or hand-drawn sketch might be useful.)

c) Now get an explicit formula for f^{-1} and check your work above.

[1] A normal line is a line that is perpendicular to the tangent line.