Probability

Math 143

Department of Mathematics and Statistics
Calvin College

Fall 2006
1 Probability Review
   - Basics
   - Random Variables
   - Mean, Variance and Standard Deviation of Random Variables

2 More Probability
   - Independence Rule
   - Generalized Sum and Product Rules
   - Working with Three or More Events
Quick Review of Probability

- Probability measures long-term relative frequency.

- Empirical: $P(E) = \frac{\# \text{ of times } E \text{ happens}}{\# \text{ of times random procedure is repeated}}$

- Theoretical: Based on mathematical assumptions and rules of probability.

- Discrete: Assign probabilities to each outcome (table works well)

- Continuous: Describe probabilities with areas under density curves
Basic Probability Rules

1. \(0 \leq P(E) \leq 1\) for any event \(E\).
2. \(P(S) = 1\), where \(S\) is the sample space.
3. \(P(E^c) = 1 - P(E)\) [so \(P(E) = 1 - P(E^c)\), too]
4. \(P(A \text{ or } B) = P(A) + P(B)\) provided \(A\) and \(B\) are mutually exclusive.

These imply a simple rule for equally likely outcomes:

\[
P(E) = \frac{\# \text{ of outcomes in } E}{\# \text{ of outcomes in } S}
\]
Random Variables

A **random variable** is a variable whose value is

- numerical, and
- determined by the outcome of some random phenomenon

A **distribution** of a random variable tells us the possible values of a random variable, and the probability of having those values. We use random variables to help us quantify the results of experiments for the purpose of analysis. Random variables are generally denoted using **capital letters**, like $X, Y, Z$. 
Examples of random variables

1. Deal a poker hand. Let $X =$ the number of face cards. $X$ assigns to $3, 7, 10, Q, K$ the value $2$.

2. Roll two six-sided dice. Let $Y =$ the sum of the two numbers rolled. $Y$ assigns to a roll of $\{2, 4\}$ the value $6$.

3. Let $H =$ the difference between the heights of a randomly selected couple (height of husband $-$ height of wife, measured in inches). If my wife and I were the randomly selected couple, then the value of $H$ would be $10.5$.

4. A family plans to have four children. Let $Y$ be the number of boys.

5. Randomly sample 100 people and ask them a yes/no question. Let $X$ be the number of people in the sample who answer yes.

There are two kinds of random variables: discrete and continuous.
Discrete Random Variables

If $X$ is discrete (takes on values from a list possibilities), we can describe the distribution of $X$ by making a table that lists each possible value of $X$ and the probability that it occurs.

<table>
<thead>
<tr>
<th>Value of $X$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$\ldots$</th>
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<tbody>
<tr>
<td>Probability</td>
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Of course, each probability ($p_i$) must be between 0 and 1 (inclusive), and the sum of all the probabilities ($\sum p_i$) must equal 1.
Example (Flipping 3 fair coins and counting heads)

Flip 3 fair coins. Let $X =$ the number of heads.

- Find the probability distribution of $X$.
- Determine $P(X \geq 2)$.
- Determine $P(X \text{ is even})$. 

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<tr>
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</tr>
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$P(X \geq 2) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$

$P(X \text{ is even}) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$
Flipping Three Fair Coins

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$P(X \geq 2) = 3/8 + 1/8 = 1/2$

$P(X$ is even) $= 1/8 + 3/8 = 1/2$
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![Probability Histogram](image-url)
Sometimes we base a model for a random variable on data collected from a sample. For example, an industrial psychologist administered a personality inventory test for passive-aggressive traits to 150 employees. Individuals were rated on a scale from 1 to 5 (passive to aggressive). The results are as follows:

<table>
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<tr>
<td>2</td>
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Let $T$ be rating of a random employee. Assuming the 150 tested employees are typical of all the employees, we can construct the probability distribution table for the random variable $T$.

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Now compute the following

- $P(T > 4) =$


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Now compute the following

- $P(T > 4) = .14$
- $P(T \geq 4) =$
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- $P(T > 4) = .14$
- $P(T \geq 4) = .14 + .20 = .34$
- $P(T \leq 2) =$
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Now compute the following:

- $P(T > 4) = .14$
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Personal Inventory

Let $T$ be rating of a random employee. Assuming the 150 tested employees are typical of all the employees, we can construct the probability distribution table for the random variable $T$.

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Suppose this test were given to 500 employees. Based on the above probability distribution, about how many would we expect to be extremely aggressive (score of 5)?
Continuous Random Variables

A continuous random variable can take all values in an interval of numbers. Therefore the probability distribution of a continuous random variable can’t be described by making a table of values and probabilities. Instead, the probability distribution of a continuous random variable $X$ is described by a density curve.

**Definition**

The probability of any event is the area under the density curve and above the values of $X$ that make up the event.

- Just like we have been drawing for normal distributions
- Use calculus to compute (or estimate) the area
- Important statistical distributions are built into software, and included in tables like the one for normal distributions
Probabilities from Density Curves

Example (Simple Density Curve)

Below is a simple density curve for a random variable $X$.

Based on the density curve, we can compute things like

$$P(X > 1) \quad P(0.5 \leq X \leq 1.5) \quad P(X = 1)$$
Recall that the normal distributions are a family of continuous probability distributions with distinctive bell-shaped density curves.

If $X$ is normally distributed with mean $\mu$ and standard deviation $\sigma$, we will write $X \sim N(\mu, \sigma)$.

If $X$ is approximately normally distributed, we will write $X \approx N(\mu, \sigma)$.

**Example (Using this notation)**

Suppose $Z \sim N(0, 1)$, then $P(-1 \leq Z \leq 1) =$
Recall that the **normal distributions** are a family of continuous probability distributions with distinctive bell-shaped density curves.

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Suppose \( Z \sim N(0, 1) \), then \( P(-1 \leq Z \leq 1) = 0.68 \) (approx.)
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**Example (Using this notation)**

Suppose \( Z \sim N(0, 1) \), then \( P(-1 \leq Z \leq 1) = 0.68 \) (approx.)

But what does it mean to take the mean or standard deviation of a random variable?
Computing the Mean from Data

When we want to compute the mean of a list of numbers \( \{x_1, x_2, \ldots, x_n\} \), we simply add the numbers and divide by how many there were.

\[
\text{mean} = \bar{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}
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\]

This can be written another way:

\[
\text{mean} = \bar{x} = \frac{x_1}{n} + \frac{x_2}{n} + \cdots + \frac{x_n}{n}
\]

\[
= \frac{1}{n} x_1 + \frac{1}{n} x_2 + \cdots + \frac{1}{n} x_n
\]

\[
= \frac{1}{n} \sum x_i
\]
Written this last way, we can figure out how to compute the mean (also called expected value) of a discrete random variable.

\[
\text{mean} = \bar{x} = x_1 \cdot \frac{1}{n} + x_2 \cdot \frac{1}{n} + \cdots + x_n \cdot \frac{1}{n} = \sum x_i \cdot \frac{1}{n}
\]

Imagine choosing one of our numbers \( \{x_1, x_2, \ldots x_n\} \) at random, each equally likely. Then \( \frac{1}{n} \) is the probability of choosing a particular number in our list. So we see that

**Key Idea**

the mean is the sum of values times probabilities

- usually call this a “weighted sum”
The Mean of a Discrete Random Variable

Suppose $X$ is a discrete random variable and the distribution of $X$ is given by

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Then to find the mean of $X$, multiply each possible value by its probability, then add all the products:

**Definition (Mean of Discrete Random Variable)**

$$\text{mean of } X = \mu_X = \sum x_i p_i$$
Example (Flipping 3 fair coins and counting heads)

Let $X$ be the random variable described by

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What is $\mu_X$?

$$0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = 1.5$$
Variance and Standard Deviation

Definition

Suppose $X$ is a discrete random variable and the distribution of $X$ is given by

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Then to find the variance of $X$, multiply the squared difference from the mean for each possible value by its probability, then add all the products:

$$\text{variance of } X = \sigma^2_X = \sum (x_i - \mu_X)^2 p_i$$

The standard deviation is the square root of the variance:

$$\sigma_X = \sqrt{\sigma^2}.$$
Flipping Three Fair Coins

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$$
\sigma^2_X = (0 - 1.5)^2 \cdot \frac{1}{8} + (1 - 1.5)^2 \cdot \frac{3}{8} + (2 - 1.5)^2 \cdot \frac{3}{8} + (3 - 1.5)^2 \cdot \frac{1}{8}
$$

$$
= 2.25 \cdot \frac{1}{8} + .25 \cdot \frac{3}{8} + .25 \cdot \frac{3}{8} + 2.25 \cdot \frac{1}{8} = 0.75
$$

$$
\sigma_X = \sqrt{.75} = 0.86603
$$
The means and standard deviations of continuous random variables are computed in a related way, but require more advanced mathematics (integral calculus) to express precisely. Basically, sums are replaced by integrals:

**Definition (You are NOT required to know this.)**

\[
\mu_X = \int xp(x) \, dx
\]

\[
(\sigma_X)^2 = \int (x - \mu_X)^2 p(x) \, dx
\]
Adding Another Rule

1. $0 \leq P(E) \leq 1$ for any event $E$.
2. $P(S) = 1$, where $S$ is the sample space.
3. $P(E^c) = 1 - P(E)$ [so $P(E) = 1 - P(E^c)$, too]
4. $P(\text{A or B}) = P(\text{A}) + P(\text{B})$ provided $\text{A}$ and $\text{B}$ are mutually exclusive.
5. $P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$ provided $\text{A}$ and $\text{B}$ are independent.

Two events are independent if knowing whether one of them occurs or not does not change the probability of the other.
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Two events are **independent** if knowing whether one of them occurs or not does not change the probability of the other.

**Example (Independent Events)**

Toss two fair coins. Getting a head on the second toss is independent of getting a head on the first toss, since the probability is still $1/2$ for getting a head on the second toss, even if we know the first toss was a head (or a tail).
A standard 6-sided die has six sides numbered 1, 2, 3, 4, 5, and 6. Each number is equally likely to be rolled (assuming a fair die).

Example (Yahtzee Probability)

In the game of Yahtzee, 5 six-sided dice are rolled. If all 5 dice have the same number, the roll is called a Yahtzee. What is the probability of rolling 5 dice and getting a Yahtzee?
An Easier Related Problem

Let’s try something easier to start:

Example (Easier)

If we roll two dice, what is the probability that the numbers rolled match? (That is, what is the probability of rolling “doubles”?)
An Easier Related Problem

Let’s try something easier to start:

Example (Easier)

If we roll two dice, what is the probability that the numbers rolled match? (That is, what is the probability of rolling “doubles”?)

Method 1: There are 36 possible rolls (for each of 6 possibilities on die 1, there are 6 possibilities on die 2). Of these only 6 are “doubles”. So the probability is \( \frac{6}{36} = \frac{1}{6} \).
Let’s try something easier to start:

**Example (Easier)**

If we roll two dice, what is the probability that the numbers rolled match? (That is, what is the probability of rolling “doubles”?)

Method 1: There are 36 possible rolls (for each of 6 possibilities on die 1, there are 6 possibilities on die 2). Of these only 6 are “doubles”. So the probability is \( \frac{6}{36} = \frac{1}{6} \).

Method 2: Change the question to “After we roll the first die, what is the probability that the second one will match it?” which has the same answer. There are 6 equally likely outcomes of the second die, only one matches, so the answer is 1/6, same as we got using method 1.
Example (original Yahtzee Probability)

In the game of Yahtzee, 5 six-sided dice are rolled. If all 5 dice have the same number, the roll is called a Yahtzee. What is the probability of rolling 5 dice and getting a Yahtzee?
Example (original Yahtzee Probability)

In the game of Yahtzee, 5 six-sided dice are rolled. If all 5 dice have the same number, the roll is called a Yahtzee. What is the probability of rolling 5 dice and getting a Yahtzee?

Method 1: Now there are $6 \times 6 \times 6 \times 6 \times 6 = 6^5 = 7776$ possible outcomes. Again only 6 are Yahtzees, so the probability is

$$\frac{6}{6^5} = \frac{1}{6^4} = \frac{1}{1296}.$$
Example (original Yahtzee Probability)

In the game of Yahtzee, 5 six-sided dice are rolled. If all 5 dice have the same number, the roll is called a Yahtzee. What is the probability of rolling 5 dice and getting a Yahtzee?

Method 1: Now there are $6 \times 6 \times 6 \times 6 \times 6 = 6^5 = 7776$ possible outcomes. Again only 6 are Yahtzees, so the probability is $\frac{6}{6^5} = \frac{1}{6^4} = \frac{1}{1296}$.

Method 2: After the first die is rolled, the probability that each subsequent die matches it is $\frac{1}{6}$. These events are independent, so we can multiply to get the probability that they all happen at once: $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{6^4} = \frac{1}{1296}$.
Mendel’s Peas

The peas from Mendel’s genetics experiments could be either yellow or green. In this simple situation, color is determined by a single gene. Each organism inherits two of these genes (one from each parent), which can be one of two types (Y,y). A pea plant produces yellow peas unless BOTH genes are type y. As a first step, Mendel bred pure line pea plants of type YY or yy. The offspring of these plants are very predictable:

- \( YY \times YY \): all offspring are \( YY \) (and therefore yellow)
- \( yy \times yy \): all offspring are \( yy \) (and therefore green)
- \( YY \times yy \): all offspring are \( Yy \) (and therefore yellow)
More interestingly, consider $Yy \times Yy$:

- $P(\text{green}) = P(yy) =$
More interestingly, consider \( Yy \times Yy \):

\[
P(\text{green}) = P(yy) = P(y \text{ from 1st } Yy \text{ and } y \text{ from 2nd } Yy) = P(y \text{ from 1st}) \cdot P(y \text{ from 2nd}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}
\]
More interestingly, consider $Yy \times Yy$:

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- $P(\text{yellow}) = 1 - P(\text{green}) = \frac{3}{4}$

$P(\text{heterozygote}) = P(Yy \text{ or } yY) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Notice where we are repeatedly using the assumptions:
- Which gene is inherited from a parent is equally likely
- The genes inherited from the parents are independent of each other

We can look at how well the data fit the theoretical probabilities to test the assumptions. (Mendel’s data fit very well.)
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More Peas

Mendel actually looked at two traits: color (yellow, green) and shape (round, wrinkled). According to his model, yellow (Y) and round were dominant (R). If these traits are inherited independently, then we can make predictions about a YyRr × YyRr cross:

- $P(\text{yellow}) = \frac{3}{4}$
- $P(\text{green}) = \frac{1}{4}$
- $P(\text{round}) = \frac{3}{4}$
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More Peas

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Example (Flipping 3 unfair coins and counting heads)

Flip 3 coins that result in heads 3/4 of the time. Let $H =$ the number of heads.

- Find the probability distribution of $H$.
- Determine $P(H \geq 2)$.
- Determine $P(H$ is even).
Flipping Three Unfair Coins

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<tbody>
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$P(H = 1) = \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = 3 \cdot \frac{3}{64} = \frac{9}{64}$
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$P(H \geq 2) = 27/64 + 27/64 = 54/64$

$P(H$ is even) $= 1/64 + 27/64 = 28/64$
What’s to Come: Generalized Probability Rules

We still need to answer the following questions:

- What is $P(A \text{ or } B)$ when $A$ and $B$ are NOT mutually exclusive?
- What is $P(A \text{ and } B)$ when $A$ and $B$ are NOT independent?
General Addition Rule

We can replace our old rule #4 with a more general version.

4. For any events $A$ and $B$,

$$P(A \text{ or } B) = \ldots$$
We can replace our old rule #4 with a more general version.

For any events $A$ and $B$,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
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Example (Rolling Dice)

If we roll two fair 6-sided dice, what is the probability of getting at least one 5?
General Addition Rule

We can replace our old rule #4 with a more general version.

4 For any events \( A \) and \( B \),

\[
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\]

Example (Rolling Dice)

If we roll two fair 6-sided dice, what is the probability of getting at least one 5?

\[
P(\text{first is 5}) = 1/6; \ P(\text{second is 5}) = 1/6 \text{ (equally likely outcomes)}
\]
\[
P(\text{both are 5}) = (1/6) \cdot (1/6) = 1/36 \text{ (independent events)}
\]
So \( P(\text{at least one is 5}) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36} \)
Conditional Probability

Consider the probability chart below which shows the probabilities for sex and favorite color in Mr. Ortez’s class.

<table>
<thead>
<tr>
<th></th>
<th>Red</th>
<th>Yellow</th>
<th>Blue</th>
</tr>
</thead>
<tbody>
<tr>
<td>boy</td>
<td>.15</td>
<td>.05</td>
<td>.25</td>
</tr>
<tr>
<td>girl</td>
<td>.10</td>
<td>.25</td>
<td>.20</td>
</tr>
</tbody>
</table>

If a student is selected at random from Mr. Ortez’s class:

- What is the probability of selecting a boy?
- What is the probability that the student’s favorite color is blue, given that the student is a girl?
- What is the probability that the student is a girl, given that the favorite color is blue?
Motivated by the preceding example, we make the following definition:

**Definition (Conditional Probability)**

\[
P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}
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**Definition (Conditional Probability)**

\[ P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} \]

Note that the following statements are equivalent:

1. **A and B are independent**
2. \( P(A \text{ and } B) = P(A) \cdot P(B) \)
3. \( P(B \mid A) = \frac{P(A) \cdot P(B)}{P(A)} = P(B) \)

So the probability of \( B \) is unchanged by knowledge of whether \( A \) occurs.
Definition (Conditional Probability)

\[ P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} \]

If we rewrite this definition, we get

**General Product Rule**

For any events \( A \) and \( B \),

\[ P(A \text{ and } B) = \frac{P(A \text{ and } B)}{P(A)} \]
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**General Product Rule**

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\[ P(A \text{ and } B) = P(A) \cdot P(B \mid A) \]
Example (Same Sex Selection)

Suppose a class has 10 men and 10 women. If three students are selected at random, what is the probability that all three are the same sex?

Let $S =$ second person is the same sex as the first.

Let $T =$ third person is the same sex as the first.

$P(S \text{ and } T) = P(S) \cdot P(T | S) = \frac{9}{19} \cdot \frac{8}{18} = \frac{4}{19}$

Note: this is less than the probability of getting the same value on a coin if we flip it three times because it 'gets harder and harder' to get someone of the same sex as we 'use them up'. (The coin probability is $\frac{1}{4}$.)
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Social science researchers have conducted extensive empirical studies and concluded that the expression

- “absence makes the heart grow fonder”
- “out of sight out of mind”

is generally true. Do you find this result surprising or not surprising?

Results (previous class):

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<th>(A) absence</th>
<th>(A^c) sight</th>
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<td>surprised ((S))</td>
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<td>5</td>
</tr>
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Q. Is “being surprised” independent of the form of the question asked?
From the Little Survey

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Is “being surprised” independent of the form of the question asked?

\[
P(S) = \frac{11}{91} = 0.12088 \quad P(A) = \frac{50}{91}
\]

\[
P(S \text{ and } A) = \frac{6}{91} = 0.0659
\]

\[
P(A) \cdot P(S) = \frac{11}{91} \cdot \frac{50}{91} = 0.0664
\]

\[
P(S|A) = \frac{6/91}{50/91} = 0.12
\]
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\[
P(S \mid A) = \frac{6/91}{50/91} = 0.12
\]
In a simple model of a recessive disease,

- each person has one of three genotypes (AA, Aa, aa) based on which of two forms each of their two genes (one from each parent) has,
- those of type aa have the disease, the others did not.

A more realistic model assigns to each of these types a probability of getting the disease.

**Definition (Penetrance)**

The following probabilities are said to describe the penetrance of the disease:

- $P(D|AA)$
- $P(D|Aa)$
- $P(D|aa)$
Penetrance of a Disease

Example (Consider the following penetrance model)

- $P(D|AA) = 0.01$
- $P(D|Aa) = 0.05$
- $P(D|aa) = 0.50$

Now consider an $AA \times aa$ cross. What is the probability that a child will have the disease?
Penetrance of a Disease

Example (Consider the following penetrance model)

- $P(D|AA) = 0.01$
- $P(D|Aa) = 0.05$
- $P(D|aa) = 0.50$

Now consider an $AA \times aa$ cross. What is the probability that a child will have the disease? $P(D) = P(D|Aa) = 0.01$ because child will be $Aa$
Penetrance of a Disease

Example (Consider the following penetrance model)

- \( P(D|AA) = 0.01 \)
- \( P(D|Aa) = 0.05 \)
- \( P(D|aa) = 0.50 \)

1. Now consider an \( AA \times aa \) cross. What is the probability that a child will have the disease?
2. In an \( AA \times Aa \) cross?
Penetrance of a Disease

Example (Consider the following penetrance model)

- \( P(D|AA) = 0.01 \)
- \( P(D|Aa) = 0.05 \)
- \( P(D|aa) = 0.50 \)

1. Now consider an \( AA \times aa \) cross. What is the probability that a child will have the disease?

2. In an \( AA \times Aa \) cross?

\[
P(D) = P(D \text{ and } AA) + P(D \text{ and } Aa) + P(D \text{ and } aa) \\
= P(AA) \cdot P(D|AA) + P(Aa) \cdot P(D|Aa) + P(aa) \cdot P(D|aa) \\
= (0.5)(0.01) + (0.5)(0.05) + (0)(0.5) = 0.03
\]
Penetrance of a Disease

Example (Consider the following penetrance model)

- \( P(D|AA) = 0.01 \)
- \( P(D|Aa) = 0.05 \)
- \( P(D|aa) = 0.50 \)

1. Now consider an \( AA \times aa \) cross. What is the probability that a child will have the disease?
2. In an \( AA \times Aa \) cross?
3. In an \( Aa \times Aa \) cross?
Penetrance of a Disease

Example (Consider the following penetrance model)

- $P(D|AA) = 0.01$
- $P(D|Aa) = 0.05$
- $P(D|aa) = 0.50$

1. Now consider an $AA \times aa$ cross. What is the probability that a child will have the disease?
2. In an $AA \times Aa$ cross?
3. In an $Aa \times Aa$ cross?

$P(D) = P(D \text{ and } AA) + P(D \text{ and } Aa) + P(D \text{ and } aa)$

$= P(AA) \cdot P(D|AA) + P(Aa) \cdot P(D|Aa) + P(aa) \cdot P(D|aa)$

$= (.25)(.01) + (.5)(.05) + (.25)(.5) = .1525$
Genetics Example 2: DMD

DMD is a serious form of Muscular Dystrophy, a sex-linked recessive disease.

- If a woman is a carrier, her sons have a 50% chance of being affected (daughters have a 50% chance of being carriers).
- 2/3 of DMD cases are inherited (from Mom); 1/3 of DMD cases are due to spontaneous mutations.

Now suppose there is a screening test such that

\[ P(T+|C+) = 0.7 \]
[**sensitivity** – how often correct when person is carrier]

\[ P(T−|C−) = 0.9 \]
[**specificity** – how often correct when person isn’t carrier]

If a woman has a child with DMD and then has a screening test done and it comes back positive, what is the probability that she is a carrier?
Genetics Example 2: DMD

Goal: Compute \( P(C+ | T+) = \frac{P(C+ \text{ and } T+)}{P(T+)} \)

Data:
- \( P(C+) = \frac{2}{3} \) [based on prior information]
- \( P(T+ | C+) = .7 \) [sensitivity]
- \( P(T- | C-) = .9 \) [specificity]
Genetics Example 2: DMD

Goal: Compute $P(C + | T +) = \frac{P(C + \text{and } T +)}{P(T +)}$

Data:
- $P(C +) = \frac{2}{3}$ [based on prior information]
- $P(T + | C +) = 0.7$ [sensitivity]
- $P(T - | C -) = 0.9$ [specificity]

Results:
- $P(T + \text{and } C +) = P(C +) \cdot P(T + | C +) = \frac{2}{3}(0.7) = \frac{14}{30}$
- $P(T + \text{and } C -) = P(C -) \cdot P(T + | C -) = \frac{1}{3}(0.1) = \frac{1}{30}$
- So $P(C + | T +) = \frac{14/30}{15/30} = \frac{14}{15} = 0.9333$

What if the screening test is negative? Compute $P(C + | T -)$ instead. Same methods work. (A. 6/15)
Genetics Example 2: DMD

Goal: Compute $P(C+ | T+)$ = $\frac{P(C+ \text{ and } T+)}{P(T+)}$

Data:
- $P(C+) = \frac{2}{3}$ [based on prior information]
- $P(T+ | C+) = 0.7$ [sensitivity]
- $P(T- | C-) = 0.9$ [specificity]

Results:
- $P(T+ \text{ and } C+) = P(C+) \cdot P(T+ | C+) = \left(\frac{2}{3}\right)(0.7) = \frac{14}{30}$
- $P(T+ \text{ and } C-) = P(C-) \cdot P(T+ | C-) = \left(\frac{1}{3}\right)(0.1) = \frac{1}{30}$
- So $P(C+ | T+)$ = $\frac{\frac{14}{30}}{\frac{15}{30}} = \frac{14}{15} = 0.9333$

What if the screening test is negative?

What if the screening test is negative?
Genetics Example 2: DMD

Goal: Compute $P(C + | T +) = \frac{P(C+ \text{ and } T+)}{P(T+)}$

Data:

- $P(C+) = \frac{2}{3}$ [based on prior information]
- $P(T + | C+) = .7$ [sensitivity]
- $P(T - | C-) = .9$ [specificity]

Results:

- $P(T + \text{ and } C+) = P(C+) \cdot P(T + | C+) = (2/3)(0.7) = 14/30$
- $P(T + \text{ and } C-) = P(C-) \cdot P(T + | C-) = (1/3)(0.1) = 1/30$
- So $P(C + | T+) = \frac{14/30}{15/30} = 14/15 = 0.9333$

What if the screening test is negative?

- Compute $P(C + | T-) \text{ instead. Same methods work. (A. 6/15)}$
You can keep your work organized on a problem like this using a table.

<table>
<thead>
<tr>
<th></th>
<th>$T+$</th>
<th>$T-$</th>
<th>total</th>
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<tbody>
<tr>
<td>$C+$</td>
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Fill in the boxes on the table as you figure them out, keeping track of the ones you need to know. In this case we want to know $P(C+|T+)$, so we need the values in the $T+$ column.
Keeping Organized: Tables

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Fill in the boxes on the table as you figure them out, keeping track of the ones you need to know. In this case we want to know $P(C+ | T+)$, so we need the values in the $T+$ column.
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$P(C+) = \frac{2}{3}$, $P(C-) = \frac{1}{3}$

Fill in the boxes on the table as you figure them out, keeping track of the ones you need to know. In this case we want to know $P(C+ | T+)$, so we need the values in the $T+$ column.
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$P(C+) = \frac{2}{3}$, $P(C-) = \frac{1}{3}$

Fill in the boxes on the table as you figure them out, keeping track of the ones you need to know. In this case we want to know $P(C+ \mid T+)$, so we need the values in the $T+$ column.

- $P(T+ \text{ and } C+) = P(C+) \cdot P(T+ \mid C+) = (2/3)(0.7) = 14/30$
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$P(C+) = \frac{2}{3}, P(C-) = \frac{1}{3}$

Fill in the boxes on the table as you figure them out, keeping track of the ones you need to know. In this case we want to know $P(C+ | T+)$, so we need the values in the $T+$ column.

- $P(T+ and C+) = P(C+) \cdot P(T+ | C+) = (2/3)(0.7) = 14/30$
- $P(T+ and C-) = P(C-) \cdot P(T+ | C-) = (1/3)(0.1) = 1/30$
- $P(T+) = P(T+ and C+) + P(T+ and C-) = 15/30$
Keeping Organized: Tables

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- $P(T+) = P(T+ and C+) + P(T+ and C-) = 15/30$
- So $P(C+ | T+) = \frac{14/30}{15/30} = 14/15 = 0.9333$
Keeping Organized: Trees

Alternatively, we can represent this information in a tree.

\[ P(C^+ \text{ and } T+) = \frac{2}{3} \cdot 0.7 \]
\[ P(C^+ \text{ and } T-) = \frac{2}{3} \cdot 0.3 \]
\[ P(C^- \text{ and } T+) = \frac{1}{3} \cdot 0.1 \]
\[ P(C^- \text{ and } T-) = \frac{1}{3} \cdot 0.9 \]
Generalized Product Rule

\[ P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B|A) \cdot P(C|A \text{ and } B) = P(A \text{ and } B) \]

Example (Red Cards)

A standard deck of cards has 52 cards (26 red; 26 black). If you are dealt 5 cards at random, what is the probability that all 5 are red?

\[ \frac{26}{52} \cdot \frac{25}{51} \cdot \frac{24}{50} \cdot \frac{23}{49} \cdot \frac{22}{48} = \frac{0.02531}{(1/2)^5} = 0.02531 \]
Generalized Product Rule

\[ P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B|A) \cdot P(C|A \text{ and } B) = P(A \text{ and } B) \]

Example (Red Cards)

A standard deck of cards has 52 cards (26 red; 26 black). If you are dealt 5 cards at random, what is the probability that all 5 are red?

\[
P(1\text{st red}) \times P(2\text{nd red} \mid 1\text{st red}) \times P(3\text{rd red} \mid 1\text{st and 2nd red}) \times P(4\text{th red} \mid 1\text{st, 2nd and 3rd are red}) \times P(5\text{th red} \mid 1\text{st, 2nd, 3rd and 4th are red}) \]

\[
= \frac{26}{52} \cdot \frac{25}{51} \cdot \frac{24}{50} \cdot \frac{23}{49} \cdot \frac{22}{48} = 0.02531 \lesssim (1/2)^5
\]
Inclusion-Exclusion Rule

\[ P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) \]

- Things in none of the events never get counted.
- Things in one event get counted once.
- Things in two events get counted twice
- Things in all three events get counted 3 times
Inclusion-Exclusion

Inclusion-Exclusion Rule

\[ P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) \]
\[ - P(A \text{ and } B) - P(A \text{ and } C) - P(B \text{ and } C) \]

- Things in none of the events never get counted.
- Things in one event get counted once.
- Things in two events get counted twice, then “uncounted” once.
- Things in all three events get counted 3 times, then “uncounted” 3 times
Inclusion-Exclusion

### Inclusion-Exclusion Rule

\[
P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) \\
- P(A \text{ and } B) - P(A \text{ and } C) - P(B \text{ and } C) \\
+ P(A \text{ and } B \text{ and } C)
\]

- Things in none of the events never get counted.
- Things in one event get \textit{counted} once.
- Things in two events get \textit{counted twice}, then \textit{“uncounted”} once.
- Things in all three events get \textit{counted 3 times}, then \textit{“uncounted” 3 times}, then \textit{counted 1 time}, for a net of one time counted.