A GEOMETRIC CHARACTERIZATION: COMPLEX ELLIPSOIDS AND THE BOCHNER-MARTINELLI KERNEL

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ABSTRACT. Boas' characterization of bounded domains for which the Bochner-Martinelli kernel is self-adjoint is extended to the case of a weighted measure. For strictly convex domains, this equivalently characterizes the ones whose Leray-Aĭzenberg kernel is self-adjoint with respect to weighted measure. In each case, the domains are complex linear images of a ball, and the measure is the Fefferman measure. The Leray-Aĭzenberg kernel for a strictly convex hypersurface in $\mathbb{C}^n$ is shown to be Möbius invariant when defined with respect to Fefferman measure.

1. INTRODUCTION

For a smooth, strictly pseudoconvex domain in complex Euclidean space, Kerzman and Stein constructed in [8] a Cauchy-Fantappiè kernel that can be used for computing and proving regularity for the Szegő projection. In one dimension, their kernel is the usual Cauchy kernel; see also their paper [7]. Their method works in part because there is a cancellation of singularities that occurs when the Cauchy-Fantappiè kernel is subtracted from its adjoint. Therefore, the kernel closely resembles the Szegő kernel.

This partly motivates the problem of determining geometric conditions under which there is complete cancellation when a kernel is subtracted from its adjoint. This involves choices for both the kernel as well as a measure on the boundary. Typically, the measure is the surface area measure obtained by identifying $\mathbb{C}^n \cong \mathbb{R}^{2n}$. For this measure, Boas proved in [2] that the Bochner-Martinelli kernel is self-adjoint for a continuously differentiable bounded domain if and only if the domain is a ball.

Here we extend that result about the Bochner-Martinelli kernel to the case of a weighted measure.

Theorem 1. *For a continuously differentiable domain $\Omega \subset\subset \mathbb{C}^n$, there is a positive measure on $\partial\Omega$ for which the Bochner-Martinelli transform is self-adjoint if and only if Ω is the image of a ball under a complex linear map.*

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Theorem 1 follows from an analogous characterization of real ellipsoids. The following proposition is proved in Section 4.

Proposition 1. *For a continuously differentiable domain $\Omega \subset\subset \mathbb{R}^N$ with normal vector $\langle \nu_1, \cdots, \nu_N \rangle$, there is a positive continuous function h defined on $\partial\Omega$ with $h(x) \sum \nu_j(x)(x_j - u_j) = h(u) \sum \nu_j(u)(u_j - x_j)$ for all $x, u \in \partial\Omega$ if and only if Ω is the image of a ball under a linear map.*

Next, restricting to twice differentiable convex domains in $\mathbb{C}^n$, we also characterize the domains whose Leray-Aĭzenberg kernel is self-adjoint with respect to weighted measure. By the Leray-Aĭzenberg kernel we mean the Cauchy-Fantappiè kernel that is constructed using supporting hyperplanes.

Theorem 2. *For a strictly convex domain $\Omega \subset\subset \mathbb{C}^n$, there is a positive measure on $\partial\Omega$ for which the Leray-Aĭzenberg transform is self-adjoint if and only if Ω is the image of a ball under a complex linear map.*

In each case, the measure is the Fefferman surface measure, which he constructed for strictly pseudoconvex domains in order for there to be a biholomorphically invariant Szegő projection. (It is a consequence of Theorem 1 that the Fefferman measure is defined for domains with self-adjoint Bochner-Martinelli transform, since complex ellipsoids are strictly pseudoconvex.) With respect to this measure, the Szegő projection and kernel satisfy a transformation law for biholomorphic maps. We show that the Leray-Aĭzenberg kernel satisfies a similar transformation law for Möbius transformations. By a Möbius transformation we mean that after embedding $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n$ in the usual way, the map is linear with respect to the homogeneous coordinates. Special cases are the complex linear maps of $\mathbb{C}^n$.

Theorem 3. *With respect to Fefferman surface measure, the Leray-Aĭzenberg transform is Möbius-invariant. Its kernel satisfies*

$$C_1(z,w) = J_F^{n/(n+1)}(z)\ C_2(F(z),F(w))\ \overline{J_F^{n/(n+1)}(w)},$$

when $F : \Omega_1 \to \Omega_2$ is a Möbius transformation and J_F is its Jacobian determinant.

It would be interesting to extend Theorem 1 to the case of a Lipschitz domain. For this we point out [11], in which M. Lim showed that among bounded Lipschitz domains, the boundary integral operator associated to the double layer potential of the Laplacian is self-adjoint only for a ball. There is a close connection between the double layer potential of the Laplacian and the Bochner-Martinelli transform. So far the author has been unable to extend Lim's method to the situation of Propositions 1 or 2 .

There also remains the question of a local version of Theorem 1. That is, for hypersurfaces $M^{2n-1} \subset \mathbb{C}^n$, if the Bochner-Martinelli kernel is self-adjoint with respect to a weighted measure, must M be contained in the Möbius image of a sphere? It would be interesting to find a differential geometric proof along the lines of the proof in [4], where the author characterized the hypersurfaces for which the skew-hermitian part of the Bochner-Martinelli kernel is less singular than usual. This result would also be an extension of Boas' local characterization of spheres and cylinders [3]. We include a proof for the case of dimension one in Section 6.

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2. The Bochner-Martinelli transform and vanishing condition

The Bochner-Martinelli and Leray-Aĭzenberg kernels are special cases of Cauchy-Fantappiè kernels. See Range [12] for a nice treatment of this larger topic. For the two examples considered here we give only a brief summary.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with continuously differentiable boundary whose outward pointing unit normal vector is $N = N_w$ at $w \in \partial\Omega$. The Bochner-Martinelli kernel is defined by

$$K(z,w) \;=\; \frac{(n-1)!}{2\pi^n}\,\frac{N_w \cdot (\overline{w}-\overline{z})}{|w-z|^{2n}} \qquad \text{for } \; w \in \partial\Omega,\; z \neq w,$$

where the dot product in the numerator means to sum the products of the complex coordinates. If $d\sigma_E$ is Euclidean surface measure, then the Bochner-Martinelli transform is the operator $f \to \mathcal{K}f$ defined for $f \in L^2(\partial\Omega)$ by

$$\mathcal{K}f(z) \;=\; \lim_{\epsilon \downarrow 0} \int_{\substack{w\in\partial\Omega \\ |z-w|>\epsilon}} f(w)K(z,w)\,d\sigma_E.$$

By the Calderón-Zygmund theory of singular integrals, the limit exists for almost all $z \in \partial\Omega$, and $\mathcal{K}$ is bounded on $L^2(\partial\Omega)$. Furthermore, the $L^2(\partial\Omega)$ adjoint of $\mathcal{K}$ has kernel $\overline{K(w,z)}$, and $\mathcal{K}$ is self-adjoint in $L^2(\partial\Omega)$ if and only if $K(z,w) = \overline{K(w,z)}$ for all $z, w \in \partial\Omega$, $z \neq w$.

If one replaces Euclidean measure with the weighted measure $h^{-1}d\sigma_E$ for some positive continuous function h on the boundary, then with respect to the new measure the transform has kernel $h(w)K(z,w)$. Furthermore, $\mathcal{K}$ is self-adjoint if and only $h(w)K(z,w) \equiv h(z)\overline{K(w,z)}$. This holds precisely when

$$h(w)N_w \cdot (\overline{w}-\overline{z}) \;=\; h(z)\overline{N}_z \cdot (z-w) \;\text{ for all }\; w,z \in \partial\Omega. \tag{1}$$

We now establish the following proposition as a consequence of Proposition 1. (Proposition 1 is proved in Section 4.) Theorem 1 then follows immediately.

Proposition 2. *For a continuously differentiable domain $\Omega \subset\subset \mathbb{C}^n$, there is a positive continuous function h defined on $\partial\Omega$ with $h(w)N_w \cdot (\overline{w}-\overline{z}) =$*

$h(z)\overline{N}_z \cdot (z-w)$ for all $w, z \in \partial\Omega$ if and only if Ω is the image of a ball under a complex linear map.

Proof of Proposition 2. For the proof of the easier direction, that is, if Ω is a complex ellipsoid then there is a positive continuous function h so that (1) is satisfied, see Section 6. There we also establish the fact already mentioned that the relevant measure for Theorems 1 and 2 is the Fefferman surface measure.

To prove the other direction, we first express (1) using two real equations by considering separately the real and imaginary parts. For this, identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ by using real coordinates $x = (x_1, \ldots, x_{2n})$ and $u = (u_1, \ldots, u_{2n})$ for w and z, with $w_j = x_j + ix_{j+n}$ and $z_j = u_j + iu_{j+n}$. If the outward pointing unit normal at $x \in \partial\Omega$ has real components $\nu_j = \nu_j(x)$, then the complex normal is $N = \langle \nu_1 + i\nu_{1+n}, \ldots, \nu_n + i\nu_{2n}\rangle$, and (1) can be written as

$$h(x) \sum_{j=1\ldots 2n} \nu_j(x)(x_j - u_j) = h(u) \sum_{j=1\ldots 2n} \nu_j(u)(u_j - x_j) \tag{2}$$

and

$$\begin{aligned} h(x) &\sum_{j=1\ldots n} [\nu_j(x)(u_{j+n} - x_{j+n}) + \nu_{j+n}(x)(x_j - u_j)] \\ &= h(u) \sum_{j=1\ldots n} [\nu_{j+n}(u)(x_j - u_j) + \nu_j(u)(u_{j+n} - x_{j+n})], \end{aligned} \tag{3}$$

for all $x, u \in \partial\Omega$. If there is a function h defined on $\partial\Omega$ so that (2) holds, then it follows from Proposition 1 that Ω must be an ellipsoid in $\mathbb{R}^{2n}$. Evidently, conditions (2) and (3) are not affected by a translation. So using the first part of the proof of Proposition 1, let us assume that Ω has defining function $r(x) = \sum_{j,k} a_{j,k} x_j x_k + R$, where $a_{j,k} = a_{k,j}, R \in \mathbb{R}$ and $h = |\nabla r|$. Then

$$h(x)\nu_j(x) \;=\; \frac{\partial r}{\partial x_j}(x) \;=\; 2 \sum_{k=1\ldots 2n} a_{j,k} x_k,$$

the second identity using the symmetry $a_{j,k} = a_{k,j}$. It follows that (3) can be written as

$$\begin{aligned} &\sum_{\substack{j=1\ldots n \\ k=1\ldots 2n}} [a_{j,k} x_k (u_{j+n} - x_{j+n}) + a_{j+n,k} x_k (x_j - u_j)] \\ &\qquad = \sum_{\substack{j=1\ldots n \\ k=1\ldots 2n}} [a_{j+n,k} u_k (x_j - u_j) + a_{j,k} u_k (u_{j+n} - x_{j+n})]. \end{aligned} \tag{4}$$

For the rest of the proof, we determine the conditions that (4) imposes on the $a_{j,k}$. Then, using these conditions, we show that r defines a complex ellipsoid.

Since we have arranged for the ellipsoid to be centered at the origin, each coordinate axis intersects $\partial\Omega$ exactly twice. Let $x = (0, \ldots, x_l, \ldots, 0)$ be a point of intersection with the lth coordinate axis, and let $u = (0, \dots, u_m, \dots, 0)$ be a point of intersection with the mth coordinate axis. Assume $1 \le l, m \le n$. Evidently $x_l, u_m \ne 0$ since the origin is inside the ellipsoid. Since $r(u) = r(-u)$, it follows that the other point of intersection with the mth coordinate axis is $-u = (0, \dots, -u_m, \dots, 0)$. Apply (4) twice, first for $x, u \in \partial\Omega$, then for $x, -u \in \partial\Omega$. Then

$$a_{l+n,l}\, x_l\, x_l - a_{m+n,l}\, x_l\, u_m \;=\; a_{l+n,m}\, u_m\, x_l - a_{m+n,m}\, u_m\, u_m,$$

and

$$a_{l+n,l}\, x_l\, x_l + a_{m+n,l}\, x_l\, u_m \;=\; -a_{l+n,m}\, u_m\, x_l - a_{m+n,m}\, u_m\, u_m.$$

Subtracting these equations gives $-2\, a_{m+n,l}\, x_l\, u_m \;=\; 2\, a_{l+n,m}\, u_m\, x_l$, which means $a_{m+n,l} = -a_{l+n,m}$ since $x_l, u_m \ne 0$. By the symmetry $a_{j,k} = a_{k,j}$, it also follows that $a_{m+n,l} = -a_{m,l+n}$.

Next, let $x = (0, \dots, x_l, \dots, 0)$ be the same point of intersection with the lth coordinate axis, but let $u = (0, \dots, u_{m+n}, \dots, 0)$ be a point of intersection with the $(m+n)$th coordinate axis. Assume $1 \le l, m \le n$. Then $u_{m+n} \ne 0$, and the other point of intersection with the $(m+n)$th coordinate axis is $-u$. As before, apply (4) twice, first for $x, u \in \partial\Omega$, then for $x, -u \in \partial\Omega$. Then

$$a_{m,l}\, x_l\, u_{m+n} + a_{l+n,l}\, x_l\, x_l \;=\; a_{l+n,m+n}\, u_{m+n}\, x_l + a_{m,m+n}\, u_{m+n}\, u_{m+n},$$

and

$$-a_{m,l}\, x_l\, u_{m+n} + a_{l+n,l}\, x_l\, x_l \;=\; -a_{l+n,m+n}\, u_{m+n}\, x_l + a_{m,m+n}\, u_{m+n}\, u_{m+n}.$$

Subtracting these equations gives $2\, a_{m,l}\, x_l\, u_{m+n} \;=\; 2\, a_{l+n,m+n}\, u_{m+n}\, x_l$. This means $a_{m,l} = a_{l+n,m+n}$ since $x_l, u_{m+n} \ne 0$. So then also $a_{m,l} = a_{m+n,l+n}$.

Finally, we express the defining function for Ω in terms of the complex coordinates $w_j = x_j + i x_{j+n}$, and we show that Ω is a complex ellipsoid. Since $x_j = (w_j + \overline{w}_j)/2$ and $x_{j+n} = (w_j - \overline{w}_j)/(2i)$, we find that

$$\begin{aligned} r(w) \;=\; & \frac{1}{4} \sum\nolimits_{j,k=1}^{n} \big[a_{j,k}\, (w_j + \overline{w}_j)(w_k + \overline{w}_k) - i\, a_{j+n,k}\, (w_j - \overline{w}_j)(w_k + \overline{w}_k) \\ & -i\, a_{j,k+n}\, (w_j + \overline{w}_j)(w_k - \overline{w}_k) - a_{j+n,k+n}(w_j - \overline{w}_j)(w_k - \overline{w}_k)\big] \;+\; R \\ =\; & \frac{1}{4} \sum\nolimits_{j,k=1}^{n} \big[(a_{j,k} - i a_{j+n,k} - i a_{j,k+n} - a_{j+n,k+n})\, w_j w_k \\ & \qquad + (a_{j,k} - i a_{j+n,k} + i a_{j,k+n} + a_{j+n,k+n})\, w_j \overline{w}_k \\ & \qquad + (a_{j,k} + i a_{j+n,k} - i a_{j,k+n} + a_{j+n,k+n})\, \overline{w}_j w_k \\ & \qquad + (a_{j,k} + i a_{j+n,k} + i a_{j,k+n} - a_{j+n,k+n})\, \overline{w}_j \overline{w}_k\big] \;+\; R. \end{aligned}$$

From the previous two paragraphs, $a_{j+n,k} = -a_{j,k+n}$ and $a_{j,k} = a_{j+n,k+n}$, so the $w_j w_k$ and $\overline{w}_j \overline{w}_k$ terms disappear. Evidently, what remains is

$$r(w) \;=\; \sum\nolimits_{j,k} (c_{j,k} w_j \overline{w}_k + \overline{c}_{j,k} \overline{w}_j w_k) + R \;=\; \sum\nolimits_{j,k} (c_{j,k} + \overline{c}_{k,j}) w_j \overline{w}_k + R,$$

where $c_{j,k} = (a_{j,k} - ia_{j+n,k} + ia_{j,k+n} + a_{j+n,k+n})/4$. If $b_{j,k} = c_{j,k} + \bar{c}_{k,j}$, then $r(w) = \sum_{j,k} b_{j,k} w_j \overline{w}_k + R$ with $b_{j,k} = \bar{b}_{k,j}$. Since Ω is assumed to be bounded, this is the defining function for a complex ellipsoid. □

3. The Leray-Aĭzenberg transform and vanishing condition

Now let $\Omega \subset\subset \mathbb{C}^n$ be a strictly convex domain with twice differentiable boundary. Let r be a defining function for Ω, so then $\Omega = \{z : r(z) < 0\}$ with $dr \neq 0$ on $\partial\Omega$. The Leray-Aĭzenberg transform is the operator defined for $f \in L^2(\partial\Omega)$ by

$$\mathcal{C}f(z) \;=\; \left(\frac{1}{2\pi i}\right)^n \int_{w\in\partial\Omega} f(w) \frac{\partial r(w) \wedge (\overline{\partial}\partial r(w))^{n-1}}{(\sum r_j(w)(w_j - z_j))^n} \quad \text{for } z \in \Omega,$$

where the derivatives in the denominator refer to the holomorphic derivatives of r; i.e., $r_j = \partial r/\partial w_j$. Similarly, $r_{\bar{j}} = \partial r/\partial \overline{w}_j$. Leray [10] showed that $\mathcal{C}$ reproduces holomorphic functions that extend continuously to the boundary.

Aĭzenberg ([1], see also [9, p.53]) extended this to the case of linearly convex domains. These are domains for which the complex tangent space never intersects the domain itself; so if $T^c_w(\partial\Omega) = \{w + v \in \mathbb{C}^n : \sum_j r_j(w) v_j = 0\}$, then $T^c_w(\partial\Omega) \subset \mathbb{C}^n \setminus \Omega$ for all $w \in \partial\Omega$. Related to this, Stanton [13] found another representation for $\mathcal{C}$ and identified a connection between its kernel and the complex Monge-Ampère equation. In Section 5, we use a similar representation in order to prove Theorem 3.

Given the convexity condition, $i^{-n}\partial r(w) \wedge (\overline{\partial}\partial r(w))^{n-1}$ is a positive multiple of Euclidean surface measure. The reason it is a real multiple can be seen by taking its complex conjugate, noting that the wedge product is anti-symmetric and $dr = \partial r + \overline{\partial} r = 0$ when restricted to $\partial\Omega$. The reason it is a positive multiple is because of the convexity hypothesis. For this we refer to Section 5 or to Stanton [13, Theorem 1].

Next, if $(\sum r_j(w)(w_j - z_j))^n = (\sum r_{\bar{j}}(z)(\bar{z}_j - \overline{w}_j))^n$ for all $w, z \in \partial\Omega$, then $\sum r_j(w)(w_j - z_j) = \sum r_{\bar{j}}(z)(\bar{z}_j - \overline{w}_j)$ for all $w, z \in \partial\Omega$. To see this, one can see from a Taylor expansion that

$$\begin{aligned} r(w) \;&=\; r(z) + \sum\nolimits_j [r_j(z)(w_j - z_j) + r_{\bar{j}}(z)(\overline{w}_j - \bar{z}_j)] + O(w - z)^2 \\ &=\; r(z) + \sum\nolimits_j [r_j(w)(w_j - z_j) - r_{\bar{j}}(z)(\bar{z}_j - \overline{w}_j)] + O(w - z)^2. \end{aligned}$$

So, a priori, $\sum r_j(w)(w_j - z_j) = \sum r_{\bar{j}}(z)(\bar{z}_j - \overline{w}_j) + O(|w - z|)^2$ for $w, z \in \partial\Omega$. Meanwhile, there is a one-parameter family of $z \in \partial\Omega$ for which $z = w + \lambda N_w$ ($\lambda \in \mathbb{C}$), and for these z there is the estimate $|\sum r_j(w)(w_j - z_j)| \approx |w - z|$. Together, these observations show that $\sum r_j(w)(w_j - z_j)$ and $\sum r_{\bar{j}}(z)(\bar{z}_j - \overline{w}_j)$ cannot differ by a nontrivial nth root of unity, and this establishes the claim.

Working formally with the kernel, then, we find that $\mathcal{C}$ is self-adjoint with respect to weighted measure on the boundary if and only if there is a positive continuous function h so that

$$(5)\qquad h(w)\sum_j r_j(w)(w_j - z_j) \;=\; h(z)\sum_j r_{\overline{j}}(z)(\overline{z}_j - \overline{w}_j) \;\text{ for all }\; w, z \in \partial\Omega.$$

The vector $(r_{\overline{1}}(w), \ldots, r_{\overline{n}}(w))$ is a multiple of the normal vector N_w, so after taking conjugates, this is the same condition as (1) for a possibly different function h. Therefore, Theorem 2 also follows from Proposition 2.

We mention that the relevant measure in this case is not connected to h in the same way as it was for the Bochner-Martinelli kernel. We also mention that the orientation on $\mathbb{C}^n$ that is used here and in Section 5 corresponds to the volume form $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$, whereas the complex structure in Section 2 takes the real coordinates in an order compatible with the volume form $dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n$. This notational difference is harmless.

4. Proof of Proposition 1

Before proving Proposition 1, we point out that if $h(x)\sum \nu_j(x)(x_j - u_j) = h(u)\sum \nu_j(u)(u_j - x_j)$ for all $x, u \in \partial\Omega$, then the function h is unique up to a constant multiple. This is because the condition can be rewritten as

$$\frac{h(x)}{h(u)} \;=\; \frac{\sum \nu_j(u)(u_j - x_j)}{\sum \nu_j(x)(x_j - u_j)}.$$

So the ratio $h(x)/h(u)$ depends only on the geometry of $\partial\Omega$ at x and u, and therefore, once the value of h is determined at one point, it is determined at all other points as well.

Then, for the easier direction of the proof we check that if $\Omega \subset \mathbb{R}^N$ is the image of a ball under a linear map, there is a function h for which

$$(6)\qquad h(x)\sum_j \nu_j(x)(x_j - u_j) \;=\; h(u)\sum_j \nu_j(u)(u_j - x_j) \;\text{ for all }\; x, u \in \partial\Omega.$$

For this, take as a defining function for Ω

$$r(x) = \sum_{j,k} a_{j,k}x_jx_k + \sum_j b_jx_j + R,$$

for constants $a_{j,k} = a_{k,j}, b_j, R \in \mathbb{R}$. (This is gotten by composing the usual defining function for a ball directly with a linear map.) Define h on $\partial\Omega$ according to $h = |\nabla r|$. Then $h(x)\nu_j(x) = r_j(x)$, where $r_j = \partial r/\partial x_j$, and we find

$$\begin{aligned}\sum_j r_j(x)(x_j - u_j) \;&=\; \sum_{j,k}(a_{j,k}x_k + a_{k,j}x_k)(x_j - u_j) + \sum_j b_j(x_j - u_j)\\ &=\; 2\,r(x) - 2R - \sum_j b_j(x_j + u_j) - \sum_{j,k}(a_{j,k} + a_{k,j})x_ku_j.\end{aligned}$$

Similarly,

$$\sum_j r_j(u)(u_j - x_j) \;=\; 2\,r(u) - 2R - \sum_j b_j(u_j + x_j) - \sum_{j,k}(a_{j,k} + a_{k,j})u_kx_j.$$

Then, since $r(x) = 0 = r(u)$ for $x, u \in \partial\Omega$, it follows that $\sum_j r_j(x)(x_j - u_j) = \sum_j r_j(u)(u_j - x_j)$, and (6) is satisfied.

Proving the converse statement is harder, but it follows the argument that Boas gave for the unweighted problem. The new idea is contained in the proof of Lemma 1, where dilations are also used in order to identify coordinates in which Ω is a ball.

The hypothesis is that there is a positive continuous function h on $\partial\Omega$ for which (6) is satisfied, and we must show that Ω is the image of a ball under a linear map. We first establish that if Ω is transformed to a new domain Ω' by a translation, rotation, or dilation, then the hypothesis also holds on Ω'. That is, there is a positive continuous function h' on $\partial\Omega'$ for which

$$(7) \quad h'(x')\sum_j \nu'_j(x')(x'_j - u'_j) \;=\; h'(u')\sum_j \nu'_j(u')(u'_j - x'_j) \;\text{ for }\; x', u' \in \partial\Omega',$$

where $\nu'(x')$ is the unit normal at x'. We treat the cases separately:

(i) If $\Omega \to \Omega'$ via $x' = x + a$ for fixed $a \in \mathbb{R}^N$, then $\nu'(x') = \nu(x)$. So if h' is defined by $h'(x') = h(x)$, then (7) follows readily from (6) since $x' - u' = x - u$ and $u' - x' = u - x$.

(ii) If $\Omega \to \Omega'$ via $x' = Ax$ for an orthogonal transformation A, then $\nu'(x') = A\nu(x)$. So if h' is defined by $h'(x') = h(x)$, then (7) follows from (6) since $\nu'(x') \cdot (x' - u') = A\nu(x) \cdot A(x - u) = \nu(x) \cdot (x - u)$, and likewise, $\nu'(u') \cdot (u' - x') = \nu(u) \cdot (u - x)$. Here, the dot product means to sum the products of the real coordinates.

(iii) From (ii), it is enough to consider a dilation in the first real direction. So take $\Omega \to \Omega'$ via $(x'_1, x'_2, \ldots, x'_N) = (\lambda x_1, x_2, \ldots, x_N)$, for $\lambda > 0$. Since $\nu'(x') = \langle \nu_1(x)/\lambda, \ldots, \nu_N(x)\rangle / [\nu_1(x)^2/\lambda^2 + \cdots + \nu_N(x)^2]^{1/2}$, we define $h'(x') = h(x)[\nu_1(x)^2/\lambda^2 + \cdots + \nu_N(x)^2]^{1/2}$. Then, (7) follows from (6) since $h'(x')\nu'(x') \cdot (x' - u') = h(x)\nu(x) \cdot (x - u)$, and likewise, $h'(u')\nu'(u') \cdot (u' - x') = h(u)\nu(u) \cdot (u - x)$. Again, the dot product means to sum the products of the real coordinates.

For Lemma 1, we use such transformations to bring Ω into a standard position. Each such step involves introducing a new function h.

In general, let e_j denote the usual Euclidean basis vector $(0, \ldots, 1, \ldots, 0)$ that has a 1 in the jth position and 0's elsewhere.

Lemma 1. *After a translation, and after making rotations and dilations in $\mathbb{R}^N$, the points $\pm e_1, +e_2, \ldots, +e_N$ lie in $\partial\Omega$, the normal vectors at these points are $\pm e_1, +e_2, \ldots, +e_N$, and $h(\pm e_1) = h(+e_2) = \cdots = h(+e_N)$.*

Proof. Start by choosing a chord of maximal length that connects two points in the boundary, and make a uniform dilation in all directions so that this chord has length 2. Following a translation and rotation, we may assume the coordinates are such that the endpoints of the chord are at $\pm e_1$. Since the chord has maximal length, it follows that the unit normal at $\pm e_1$ is $\pm e_1$.

We first apply (6) using $x = +e_1$, $\nu(x) = +e_1$ and $u = -e_1$, $\nu(u) = -e_1$. This leads to $h(e_1)(1)(1-(-1)) = h(-e_1)(-1)(-1-1)$. So $h(e_1) = h(-e_1)$.

We then proceed by induction on $k = 1, \dots, N$. As the *inductive hypothesis*, suppose that coordinates have been chosen so that points $\pm e_1, +e_2, \dots, +e_k$ lie in $\partial\Omega$ and their respective unit normals are $\pm e_1, +e_2, \dots, +e_k$. Moreover, suppose that $h(\pm e_1) = h(+e_2) = \cdots = h(+e_k)$. The base case of the induction (with $k = 1$) has already been established.

So suppose the inductive hypothesis is true for $1 \le k < N$, and let b be a point in $\partial\Omega$ that is maximally distant from the k-dimensional hyperplane spanned by vectors $e_1, \dots, e_k$. Necessarily the normal vector at b is perpendicular to this hyperplane, and after a rotation in the remaining directions $e_{k+1}, \dots, e_N$, we may assume this vector is e_{k+1}, and the point b can be expressed as $b = (b_1, \dots, b_{k+1}, 0, \dots, 0)$. Following a dilation in the direction e_{k+1}, we may assume that $b_{k+1} = 1$. Notice that the rotation and dilation only involve the remaining directions, and therefore preserve the inductive hypothesis. That is, they preserve points $\pm e_1, e_2, \dots, e_k \in \partial\Omega$, along with the respective normal vectors $\pm e_1, e_2, \dots, e_k$, and the value of h at these points. (See (ii) and (iii) above.) We therefore continue to call the domain Ω and the associated function h.

Now apply (6) twice, first using $x = +e_1$, $\nu(x) = +e_1$ and $u = b$, $\nu(u) = +e_{k+1}$, and then using $x = -e_1$, $\nu(x) = -e_1$ and $u = b$, $\nu(u) = +e_{k+1}$. Then

$$h(+e_1)(+1)(+1 - b_1) = h(b)(1)(1-0),$$

and

$$h(-e_1)(-1)(-1 - b_1) = h(b)(1)(1-0).$$

If the two equations are added, one finds that $h(b) = h(\pm e_1)$, since $h(+e_1) = h(-e_1)$. Likewise, if the equations are subtracted, one finds that $b_1 = 0$, since $h(+e_1) = h(-e_1) \neq 0$.

So far, the argument already shows that if the inductive hypothesis holds for $k = 1$, then it also holds for $k = 2$. So suppose $k > 2$, and continue with the same argument. For $2 \le j < k$, apply (6) using $x = +e_j$, $\nu(x) = +e_j$ and $u = b$, $\nu(u) = e_{k+1}$. Then,

$$h(+e_j)(1)(1 - b_j) = h(b)(1)(1-0),$$

and as already $h(b) = h(\pm e_1) = h(e_j) \neq 0$, it follows that $b_j = 0$. It then follows that $b = e_{k+1}$. Putting everything together, we have shown that $e_{k+1} \in \partial\Omega$, the normal vector there is e_{k+1}, and $h(e_{k+1}) = h(\pm e_1)$. So the inductive hypothesis also holds for $k+1$, and the lemma is proved. ☐

Lemma 2. *The points $-e_2, \dots, -e_N$ then also lie in $\partial\Omega$, their respective normal vectors are $-e_2, \dots, -e_N$, and $h(\pm e_1) = h(-e_2) = \cdots = h(-e_N)$.*

Proof. For $2 \le k \le N$, suppose the line spanned by e_k also intersects $\partial\Omega$ at the point $c = (0, \dots, c_k, \dots, 0)$. There is at least one such point c, and then $c_k \neq 1$. Suppose also that the outward pointing unit normal at c is $\nu(c) = \langle \nu_1, \dots, \nu_N \rangle$.

Then apply (6) twice, first using $x = +e_1$, $\nu(x) = +e_1$ and $u = c$, $\nu(u) = \langle \nu_1, \dots, \nu_N \rangle$, and then using $x = -e_1$, $\nu(x) = -e_1$ and $u = c$, $\nu(u) = \langle \nu_1, \dots, \nu_N \rangle$. Then,

$$h(+e_1)(+1)(+1-0) = h(c)\,[\nu_1(0-1) + \nu_k(c_k - 0)]$$

and

$$h(-e_1)(-1)(-1-0) = h(c)\,[\nu_1(0+1) + \nu_k(c_k - 0)].$$

Since $h(+e_1) = h(-e_1)$, adding the two equations gives $h(e_1) = h(c)\nu_k c_k$. Since $h(e_1) \neq 0$, this also means that $\nu_k, c_k \neq 0$. Subtracting the two equations gives $0 = h(c) \cdot \nu_1$, which means $\nu_1 = 0$ since $h(c) \neq 0$.

Next apply (6) using $x = +e_k$, $\nu(x) = +e_k$ and $u = c$, $\nu(u) = \langle \nu_1, \dots, \nu_N \rangle$. Then,

$$h(e_k)(1)(1 - c_k) = h(c)\nu_k(c_k - 1).$$

As $c_k \neq 1$, it follows that $h(e_k) = -h(c)\nu_k$, which together with $h(e_1) = h(c)\nu_k c_k$ from the previous paragraph, leads to $-h(c)\nu_k = h(c)\nu_k c_k$, since $h(e_k) = h(e_1)$. As $h(c) \neq 0$, and as $\nu_k \neq 0$ from the previous paragraph, it follows that $c_k = -1$. So in fact, $c = -e_k \in \partial\Omega$.

So suppose that $2 \leq j \leq N$ with $j \neq k$, and apply (6) using $x = +e_j$, $\nu(x) = +e_j$ and $u = c = -e_k$, $\nu(u) = \langle 0, \nu_2, \dots, \nu_N \rangle$. Then,

$$h(+e_j)(+1)(+1-0) = h(-e_k)\,[\nu_j(0-1) + \nu_k(-1-0)].$$

Since $h(e_j) = h(e_1)$, this means $h(e_1) = h(-e_k)(-\nu_j - \nu_k)$. But following the previous paragraph, we already have $h(e_1) = -h(-e_k)\nu_k$. So $0 = h(-e_k)\nu_j$, and it follows that $\nu_j = 0$ since $h(-e_k) \neq 0$.

We conclude that $\nu_j = 0$ for each $1 \leq j \leq N$ except $j = k$, and since $\langle \nu_1, \dots, \nu_N \rangle$ has unit length, we find that $\nu_k = \pm 1$. Then, since $h(e_1) = -h(-e_k)\nu_k$, and since both $h(e_1)$ and $h(-e_k)$ must be positive, we conclude first that $\nu_k = -1$, and then $h(-e_k) = h(e_1)$. So the lemma is proved. $\square$

By the two lemmas, then, after making translations, rotations, and dilations, the points $\pm e_1, \dots, \pm e_N$ all lie in $\partial\Omega$, their respective normal vectors are $\pm e_1, \dots, \pm e_N$, and $h(\pm e_1) = \cdots = h(\pm e_N)$. Now let $x = (x_1, \dots, x_N)$ be an arbitrary point of $\partial\Omega$, and suppose its normal vector is $\langle \nu_1, \dots, \nu_N \rangle$. Apply (6) twice, first using $x = x$, $\nu(x) = \langle \nu_1, \dots, \nu_N \rangle$ and $u = +e_j$, $\nu(u) = +e_j$, and then using $x = x$, $\nu(x) = \langle \nu_1, \dots, \nu_N \rangle$ and $u = -e_j$, $\nu(u) = -e_j$. Then,

$$h(x)\,\nu(x) \cdot (x - e_j) = h(+e_j)\,(+e_j) \cdot (+e_j - x),$$

and

$$h(x)\,\nu(x) \cdot (x + e_j) = h(-e_j)\,(-e_j) \cdot (-e_j - x),$$

where the dot product means to sum the products of the real coordinates. Since $h(-e_j) = h(+e_j)$, adding the equations gives $h(x)\,\nu(x) \cdot x = h(e_j)$. Subtracting them gives $h(x)\,\nu(x) \cdot e_j = h(e_j)\,e_j \cdot x$, so $h(x)\nu_j = h(e_j)x_j$.

Multiplying this last equation by x_j and summing on $1 \le j \le N$ leads to $h(x)\,\nu(x)\cdot x = h(e_j)\,x\cdot x$. But already $h(x)\,\nu(x)\cdot x = h(e_j)$ so then $x\cdot x = 1$. This means that x lies on the unit sphere; in fact, $\partial\Omega$ must be the unit sphere. In these coordinates we also have $\nu(x) = x$, so $h(x) = h(e_j)$ and h is constant.

Since Ω is a ball after making the transformations, then it must be an ellipsoid before the transformations, and the proposition is proved.

5. Möbius invariance of the Leray-Aĭzenberg kernel

In this section we prove Theorem 3. That is, if the Leray-Aĭzenberg kernel is defined with respect to Fefferman measure, then under Möbius transformation it satisfies the same transformation law as the Szegő kernel, provided the Szegő kernel is also defined with respect to Fefferman measure.

The Fefferman surface measure $d\sigma_F$ for a convex domain $\Omega = \{z : r(z) < 0\}$ can be defined as the measure on $\partial\Omega$ for which

$$d\sigma_F \wedge dr = -i^n\,2^{-n+1} J(r)^{1/(n+1)} dz_1 \wedge \overline{dz}_1 \wedge \cdots \wedge dz_n \wedge \overline{dz}_n$$
$$= -2\,J(r)^{1/(n+1)} dV,$$

where $J(r)$ is the Levi determinant

$$J(r) \;=\; -1\cdot\det\begin{pmatrix} r & r_{\overline{k}} \\ r_j & r_{j\overline{k}} \end{pmatrix}.$$

We explain briefly why $J(r)$ is positive, which is needed for $J(r)^{1/(n+1)}$ to be well-defined. (See also Range [12, p.288-289].) If Ω is strictly convex, then the $n\times n$ matrix $(r_{j\overline{k}})$ is hermitian and has positive eigenvalues. At any point $w\in\partial\Omega$, one can make a unitary change of coordinates so that this matrix is diagonalized at w and has positive entries along the diagonal. In these coordinates, one can compute directly that $J(r)$ is positive at w so long as $dr \ne 0$. This is true for any point $w\in\partial\Omega$, so $J(r) > 0$ on $\partial\Omega$.

Any constant multiple of $d\sigma_F$ will also be an invariant surface measure. Here, the constant was chosen so that the measure coincides with the Euclidean surface measure for a sphere as well as the arclength measure in dimension one. Fefferman introduced this measure in [5, p.259].

With respect to this measure, the Leray-Aĭzenberg kernel is given by

$$C(z,w) \;=\; \frac{(n-1)!}{2\pi^n}\,\frac{J(r)^{n/(n+1)}(w)}{\left(\sum r_j(w)(w_j - z_j)\right)^n}.$$

This follows since then

$$C(z,w)\,d\sigma_F\wedge dr \;=\; -\frac{(n-1)!}{\pi^n}\,\frac{J(r)(w)\,dV}{\left(\sum r_j(w)(w_j - z_j)\right)^n},$$

and

$$\frac{1}{(2\pi i)^n}\,\frac{\partial r(w)\wedge(\overline{\partial}\partial r(w))^{n-1}}{\left(\sum r_j(w)(w_j - z_j)\right)^n}\wedge dr \;=\; \frac{1}{(2\pi i)^n}\,\frac{\partial r(w)\wedge\overline{\partial} r(w)\wedge(\overline{\partial}\partial r(w))^{n-1}}{\left(\sum r_j(w)(w_j - z_j)\right)^n}.$$

The last expressions are equal, as $\partial r\wedge\overline{\partial}r\wedge(\overline{\partial}\partial r)^{n-1}=-(2i)^n(n-1)!J(r)dV$. Stanton proved an identity equivalent to this one in [13, Theorem 1], for instance. We remark that the last identity also shows

$$dr\wedge(i^{-n}\partial r\wedge(\overline{\partial}\partial r)^{n-1})=2^n(n-1)!\,J(r)\,dV,$$

so $i^{-n}\partial r\wedge(\overline{\partial}\partial r)^{n-1}$ is a positive multiple of surface measure, since $J(r)>0$. This fact was mentioned in Section 3.

We are then able to prove Theorem 3, that is, under Möbius transformations, the Leray-Aĭzenberg kernel satisfies the transformation law

$$C_1(z,w)=J_F^{n/(n+1)}(z)\;C_2(F(z),F(w))\;\overline{J_F^{n/(n+1)}(w)}. \tag{8}$$

For the Szegő kernel (defined with respect to Fefferman measure) this transformation law is proved in Hirachi [6, (4.1)].

The proof uses the following facts. Suppose $F=(f^1,\dots f^n):\mathbb{C}^n\to\mathbb{C}^n$ is a Möbius transformation with $f^j=g_j/g_{n+1}$ $(1\le j\le n)$, $g_j(w)=a_{j,1}w_1+\cdots+a_{j,n}w_n+a_{j,n+1}$ $(1\le j\le n+1)$, and $\det(a_{j,k})=1$. Recall that $J_F=\det F'$. Then,

(a) $J(r\circ F)(z)\;=\;J(r)(F(z))\cdot|J_F(z)|^2$

(b) $J_F(z)\;=\;(1/g_{n+1}(z))^{n+1}$

(c) $\sum(r\circ F)_j(w)(w_j-z_j)$
$\qquad=\sum r_j(F(w))(f^j(w)-f^j(z))g_{n+1}(z)/g_{n+1}(w).$

Only (b) and (c) require that F is a Möbius transformation; (a) is also true for general biholomorphic maps. The statement of (a) seems to be well-known, though the author is unaware of a convenient reference for it. Its proof follows by first expressing the chain rule in the matrix form

$$\begin{pmatrix} r\circ F & (r\circ F)_{\overline{k}} \\ (r\circ F)_j & (r\circ F)_{j\overline{k}}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & f_j^l\end{pmatrix}\begin{pmatrix} r & r_{\overline{m}}\\ r_l & r_{l\overline{m}}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & f_{\overline{k}}^{\overline{m}}\end{pmatrix},$$

then taking determinants of both sides. (In this equation, all subscripts refer to derivatives.) For (b),

$$\begin{aligned} J_F(z)\;&=\;\det\left(\frac{\partial f^j}{\partial z_k}\right)\;=\;\frac{1}{g_{n+1}^n}\det\left(a_{j,k}-a_{n+1,k}\frac{g_j}{g_{n+1}}\right)\\ &=\;\frac{1}{g_{n+1}^n}\det\begin{pmatrix} a_{j,k}-a_{n+1,k}\,g_j/g_{n+1} & 0\\ a_{n+1,k} & 1\end{pmatrix}\\ &=\;\frac{1}{g_{n+1}^n}\det\begin{pmatrix} a_{j,k} & g_j/g_{n+1}\\ a_{n+1,k} & 1\end{pmatrix}\;=\;\frac{1}{g_{n+1}^{n+1}}\det\begin{pmatrix} a_{j,k} & g_j\\ a_{n+1,k} & g_{n+1}\end{pmatrix}\\ &=\;\frac{1}{g_{n+1}^{n+1}}\begin{pmatrix} a_{j,k} & a_{j,n+1}\\ a_{n+1,k} & a_{n+1,n+1}\end{pmatrix}\;=\;\frac{1}{g_{n+1}(z)^{n+1}}.\end{aligned}$$

For (c),

$$\sum_{k=1\ldots n} (r\circ F)_k(w)(w_k - z_k)$$
$$= \sum_{j,k=1\ldots n} r_j(F(w))\, g_{n+1}(w)^{-1}\left[a_{j,k} - a_{n+1,k}\, g_j(w)/g_{n+1}(w)\right](w_k - z_k)$$
$$= \sum_{j=1\ldots n} r_j(F(w))\, g_{n+1}(w)^{-1}\left[g_j(w) - g_j(z) - [g_{n+1}(w) - g_{n+1}(z)]\,\frac{g_j(w)}{g_{n+1}(w)}\right]$$
$$= \sum_{j=1\ldots n} r_j(F(w))\, g_{n+1}(w)^{-1}\left[g_{n+1}(z)\, g_j(w)/g_{n+1}(w) - g_j(z)\right]$$
$$= \sum_{j=1\ldots n} r_j(F(w))(f^j(w) - f^j(z))g_{n+1}(z)/g_{n+1}(w).$$

Proof of (8). So suppose that $F : \Omega_1 \to \Omega_2$ is a Möbius transformation, C_j is the Leray-Aĭzenberg kernel for Ω_j, and r is a defining function for Ω_2. Then $r\circ F$ is a defining function for Ω_1, and

$$C_1(z,w)\cdot 2\pi^n/(n-1)! = \frac{J(r\circ F)^{n/(n+1)}(w)}{\left(\sum(r\circ F)_j(w)(w_j - z_j)\right)^n}$$
$$= \frac{J(r)^{n/(n+1)}(F(w))\cdot |J_F(w)|^{2n/(n+1)}}{\left(\sum(r\circ F)_j(w)(w_j - z_j)\right)^n}$$
$$= \frac{J(r)^{n/(n+1)}(F(w))}{\left(\sum r_j(F(w))(f^j(w) - f^j(z))\right)^n}\cdot\frac{g_{n+1}(w)^n}{g_{n+1}(z)^n}\cdot |J_F(w)|^{2n/(n+1)}$$
$$= J_F^{n/(n+1)}(z)\cdot\frac{J(r)^{n/(n+1)}(F(w))}{\left(\sum r_j(F(w))(f^j(w) - f^j(z))\right)^n}\cdot\overline{J_F^{n/(n+1)}(w)}$$
$$= J_F^{n/(n+1)}(z)\; C_2(F(z),F(w))\;\overline{J_F^{n/(n+1)}(w)}\cdot 2\pi^n/(n-1)!.$$

This establishes (8), so Theorem 3 is proved. ∎

6. Examples

6.1. The image of a ball under a general complex linear map has defining function

$$r(w) = \sum_{j,k} a_{j,k} w_j \overline{w}_k + \sum_j (b_j w_j + \overline{b}_j \overline{w}_j) + R,$$

for constants $a_{j,k} = \overline{a_{k,j}}, b_j \in \mathbb{C}$ and $R \in \mathbb{R}$. This is gotten by composing the defining function $\tilde{r}(w) = |w-\alpha|^2 - 1$ for a unit ball directly with a complex linear map F, say $F(w) = A\cdot w$ for $A \in M_{n,n}(\mathbb{C})$, so that $r = \tilde{r}\circ F$. Then, for $w, z \in \partial\Omega$,

$$\sum\nolimits_j r_{\overline{j}}(w)(\overline{w}_j - \overline{z}_j) = r(w) - \sum\nolimits_{j,k} a_{j,k}\, w_j \overline{z}_k - \sum\nolimits_j (b_j w_j + \overline{b}_j \overline{z}_j) - R$$

and

$$\sum\nolimits_j r_j(z)(z_j - w_j) = r(z) - \sum\nolimits_{j,k} a_{j,k}\,\overline{z}_k w_j - \sum\nolimits_j (\overline{b}_j \overline{z}_j + b_j w_j) - R.$$

Since $r(w) = 0 = r(z)$ for $w, z \in \partial\Omega$, it follows that $\sum r_{\overline{j}}(w)(\overline{w}_j - \overline{z}_j) = \sum r_j(z)(z_j - w_j)$. Then (1) is satisfied provided h is defined by $h = |\nabla r|$, since then $\langle r_{\overline{1}}(w), \ldots, r_{\overline{n}}(w)\rangle = \frac{1}{2}h(w)N_w$.

Moreover, it follows that the Bochner-Martinelli transform is self-adjoint with respect to the weighted measure $h^{-1}d\sigma_E$ where $d\sigma_E$ is the Euclidean surface measure. We claim that the weighted measure $h^{-1}d\sigma_E$ is a constant multiple of the Fefferman measure. From the definition of h it follows that $(h^{-1}d\sigma_E) \wedge dr$ is a constant multiple of dV. Moreover, a simple computation shows $J(\tilde{r})$ is constant. So then $J(r)$ is also constant, using (a) from the previous section and the fact that F is linear. It then follows from the defining expression for $d\sigma_F$ that $d\sigma_F \wedge dr$ is also a constant multiple of dV. From this we conclude that $d\sigma_F$ is a constant multiple of $h^{-1}d\sigma_E$, and this establishes the claim.

Now consider the Leray-Aĭzenberg kernel defined with respect to Fefferman measure, as in Section 5. Then, using the remark from Section 3 about taking nth roots, and since $J(r)$ is constant, it follows that

$$C(z,w) \equiv \overline{C(w,z)} \text{ if and only if } \sum r_j(w)(w_j - z_j) \equiv \sum r_{\overline{j}}(z)(\overline{z}_j - \overline{w}_j).$$

So, for bounded convex domains, the Leray-Aĭzenberg kernel too is self-adjoint only for complex ellipsoids, and only when defined using Fefferman measure.

6.2. For $n = 1$, Theorem 1 says that the Cauchy transform is self-adjoint if and only if Ω is a disc, and in this case the measure must be a (constant) multiple of the arclength measure. This can also be proved directly using Taylor expansions. For instance, suppose $\partial\Omega$ is three times differentiable and $z = z(s)$ gives $\partial\Omega$ an arclength parameterization. Let $h = h(s)$ be a differentiable function. The Cauchy transform with respect to the arclength measure has kernel $C(t,s) = (2\pi i)^{-1}z'(s)/(z(s) - z(t))$. So with respect to the weighted measure $h^{-1}ds$, the transform is self-adjoint if and only if

$$\frac{z'(s)h(s)}{z(s) - z(t)} = \frac{\overline{z}'(t)h(t)}{\overline{z}(s) - \overline{z}(t)} \quad \text{for } s \neq t.$$

Using a Taylor expansion at the diagonal, one finds that

$$z'(s)h(s)\,(\overline{z}(s) - \overline{z}(t)) - \overline{z}'(t)h(t)\,(z(s) - z(t)) = h'(s)(t-s)^2 + o(t-s)^2,$$

so if the transform is self-adjoint, then h must be constant. In that case, another expansion shows that

$$z'(s)\,(\overline{z}(s) - \overline{z}(t)) - \overline{z}'(t)\,(z(s) - z(t)) = -i\kappa'(s)(t-s)^3/6 + o(t-s)^3,$$

where $\kappa(s)$ is the curvature of $\partial\Omega$ at $z(s)$. So the Cauchy transform can only be self-adjoint with respect to arclength measure, and only if the boundary has constant curvature. This establishes the local version of Theorem 1 in the case of dimension one.

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