THE MÖBIUS GEOMETRY OF HYPERSURFACES, II

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1. Introduction

Let r be a defining function for a twice differentiable real hypersurface $M^{2n-1} \subset \mathbb{C}^n$ near $p \in M$. It is a familiar fact in several complex variables that the Levi determinant,

$$\mathcal{L}_{r,p} = -\det \begin{pmatrix} r & \frac{\partial r}{\partial \overline{z}_1} & \cdots & \frac{\partial r}{\partial \overline{z}_n} \\ \frac{\partial r}{\partial z_1} & \frac{\partial^2 r}{\partial z_1 \partial \overline{z}_1} & \cdots & \frac{\partial^2 r}{\partial z_1 \partial \overline{z}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial r}{\partial z_n} & \frac{\partial^2 r}{\partial z_n \partial \overline{z}_1} & \cdots & \frac{\partial^2 r}{\partial z_n \partial \overline{z}_n} \end{pmatrix},$$

obeys a transformation law under biholomorphism. If r is normalized, this determinant can be interpreted as the hermitian part of the Gaussian curvature of M. As suggested in [5], there corresponds a law for what might be interpreted as the non-hermitian part of the Gaussian curvature,

$$\mathcal{Q}_{r,p} = -\det \begin{pmatrix} r & \frac{\partial r}{\partial z_1} & \cdots & \frac{\partial r}{\partial z_n} \\ \frac{\partial r}{\partial z_1} & \frac{\partial^2 r}{\partial z_1 \partial z_1} & \cdots & \frac{\partial^2 r}{\partial z_1 \partial z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial r}{\partial z_n} & \frac{\partial^2 r}{\partial z_n \partial z_1} & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n} \end{pmatrix},$$

provided the biholomorphism is a Möbius transformation. Combining these rules, the quotient $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p}$ behaves like a Möbius invariant curvature function, assuming M is Levi non-degenerate.

In this paper we prove the following.

Theorem 1. *Let $M^3 \subset \mathbb{C}^2$ be a non–Levi-flat, three times differentiable hypersurface, and suppose there is a constant $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \neq 0, 1$ so that for all $p \in M$,*

$$\mathcal{Q}_{r,p} = \varepsilon\, \mathcal{L}_{r,p}. \qquad (1)$$

Then M is contained in the image of

$$M_\varepsilon \stackrel{\text{def}}{=} \{(z_1, z_2) : (z_1 + \overline{z}_1) + |z_2|^2 + \operatorname{Re}(\varepsilon z_2^2) = 0\}$$

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under an affine map of the form $F(z) = Az + b$ where $0 \neq \det A \in \mathbb{R}$.

The converse of Theorem 1 is true, too, and is easily proved. It is important to note that condition (1) does not depend on the choice of the defining function. In addition, Chris Hammond has observed that the surfaces M_ε are in fact homogeneous with respect to the group of affine transformations described in Theorem 1. For more on this, and related questions, see his thesis [6].

Related to the determinants $\mathcal{L}_{r,p}$ and $\mathcal{Q}_{r,p}$ are the quadratic forms, defined for $s, t \in \mathbb{C}^n$, by

$$L_{r,p}(s, \bar{t}) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p)\, s_j \bar{t}_k \quad \text{and} \quad Q_{r,p}(s, t) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(p)\, s_j t_k.$$

These, too, transform under biholomorphism and Möbius transformation, respectively, when restricted to the complex tangent space. ($L_{r,p}$ is the Levi form.) Earlier, the author addressed the case $\varepsilon = 0$ and proved the following.

Theorem 2 ([5])**.** *Suppose that $M^{2n-1} \subset \mathbb{C}^n$ is a non–Levi-flat, three times differentiable hypersurface and that, for all $p \in M$,*

$$Q_{r,p}(s, s) = 0 \ \text{ for } \ s = (s_1, \ldots, s_n) \ \text{ with } \ \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p) s_j = 0. \tag{2}$$

Then M is contained in a hermitian quadric surface in $\mathbb{C}^n$.

In dimension two, the determinants $\mathcal{L}_{r,p}$ and $\mathcal{Q}_{r,p}$ coincide with the quantities $L_{r,p}(s, \bar{s})$ and $Q_{r,p}(s, s)$ where s is the special complex tangential direction $(-\partial r/\partial z_2,\, \partial r/\partial z_1)$. This means that condition (1) can be rewritten $Q_{r,p}(s, s) = \varepsilon\, L_{r,p}(s, \bar{s})$, and this reduces to condition (2) when $\varepsilon = 0$. In this way, Theorem 1 generalizes Theorem 2 to nonzero ε for the case $n = 2$.

It would be an interesting problem to extend Theorem 1 further by considering dimensions higher than two. For this, it would presumably be necessary to put restrictions on the eigenvalues of some combination of the forms $Q_{r,p}$ and $L_{r,p}$, rather than just work with the determinants $\mathcal{Q}_{r,p}$ and $\mathcal{L}_{r,p}$.

In [3] the author proved that the Leray transform is invariant under Möbius transformation, provided it is defined with respect to Fefferman measure. (For a convex surface, the Leray transform is the Cauchy-Fantappiè operator whose kernel is constructed using supporting complex hyperplanes.) So another interesting problem would be to estimate the norm of this transform using quantities derived from $|\mathcal{Q}/\mathcal{L}|$. In particular, for the surface M_ε, it would be good to know how the norm of the Leray transform depends on $|\varepsilon|$. This also would extend to higher dimensions the author's result [4] that describes how the spectrum of the Kerzman-Stein operator depends on the eccentricity of an ellipse.

In this direction, we point out Barrett and Lanzani's recent work [2] on the Leray transform for convex Reinhardt domains in $\mathbb{C}^2$. They establish L^2 regularity and compute essential spectra for this transform taken with respect to a family of boundary measures that includes surface measure. A special case is the set of L^p balls, which too have constant $|\mathcal{Q}/\mathcal{L}|$, though here the absolute values are necessary. We also mention Barrett's work [1] that gives a careful description of Möbius invariant geometry in one and several variables especially as it pertains to the Cauchy and Leray transforms.

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2. Möbius invariance of $\mathcal{Q}_{r,p}$ in $\mathbb{C}^n$

In this section we establish transformation formulas for $\mathcal{L}_{r,p}$ and $\mathcal{Q}_{r,p}$, we show how the proof of Theorem 1 can be reduced to the case $\varepsilon \in \mathbb{R}^+ \setminus \{1\}$, and we prove that condition (1) is independent of the choice of defining function.

By way of definition, a Möbius transformation on $\mathbb{C}^n$ is a fractional linear transformation. Specifically, a Möbius transformation is a function $F = (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ where $f_j = g_j/g_{n+1}$,

$$g_j(z) = a_{j,1}z_1 + \cdots + a_{j,n}z_n + a_{j,n+1},$$

and $\det(a_{j,k})_{j,k=1,\ldots,n+1} = 1$. The condition $\det(a_{j,k}) = 1$ acts as a normalization and has no effect the transformation itself.

Algebraically, these transformations form a group that acts on $\mathbb{C}^n$ and is isomorphic to $SL_{n+1}(\mathbb{C})$. In particular, if $\mathbb{C}^n$ is embedded in $\mathbb{CP}^n$ in the usual way, then they can be viewed as linear transformations in the homogeneous coordinates.

The affine transformations that are described in Theorem 1 are exactly the subgroup of Möbius transformations for which $\det F'$ is real. For such maps it is necessary (but not sufficient) that g_{n+1} is constant.

The following result is completely analogous to Proposition 2 in [5], where the biholomorphic and Möbius invariance of the forms $L_{r,p}$ and $Q_{r,p}$ was verified.

Proposition 1. *Let $M^{2n-1} \subset \mathbb{C}^n$ be a twice differentiable hypersurface near $p \in M$ and let $w = F(z)$ be biholomorphic in a neighborhood U of p. If $r \in C^2(U)$ is a defining function for M near p, then $M' = F(M \cap U)$ is twice differentiable, it has defining function $r \circ F^{-1}$ near $F(p)$, and*

$$\mathcal{L}_{r,p} = \mathcal{L}_{r\circ F^{-1},F(p)} \cdot |\det F'(p)|^2. \tag{3}$$

Furthermore, if F is a Möbius transformation, then

$$\mathcal{Q}_{r,p} = \mathcal{Q}_{r\circ F^{-1},F(p)} \cdot (\det F'(p))^2. \tag{4}$$

Proof. Suppose that $F = (f_1, \dots, f_n)$. Then using the chain rule, expressed in matrix form,

$$\begin{pmatrix} r & \dfrac{\partial r}{\partial \overline{z}_k} \\ \dfrac{\partial r}{\partial z_j} & \dfrac{\partial^2 r}{\partial z_j \partial \overline{z}_k} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{\partial f_l}{\partial z_j} \end{pmatrix} \begin{pmatrix} r \circ F^{-1} & \dfrac{\partial (r \circ F^{-1})}{\partial \overline{w}_m} \\ \dfrac{\partial (r \circ F^{-1})}{\partial w_l} & \dfrac{\partial^2 (r \circ F^{-1})}{\partial w_l \partial \overline{w}_m} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \overline{\dfrac{\partial f_m}{\partial z_k}} \end{pmatrix},$$

where the partial derivatives are evaluated at p or $F(p)$ as appropriate. After taking the determinant of both sides, identity (3) is proved.

It also follows from the chain rule, applied individually to the partial derivatives, that

$$(5) \quad \begin{pmatrix} r & \dfrac{\partial r}{\partial z_k} \\ \dfrac{\partial r}{\partial z_j} & \dfrac{\partial^2 r}{\partial z_j \partial z_k} \end{pmatrix}$$

$$= \begin{pmatrix} r \circ F^{-1} & \displaystyle\sum_m \frac{\partial (r \circ F^{-1})}{\partial w_m} \frac{\partial f_m}{\partial z_k} \\ \displaystyle\sum_l \frac{\partial (r \circ F^{-1})}{\partial w_l} \frac{\partial f_l}{\partial z_j} & \displaystyle\sum_{l,m} \frac{\partial^2 (r \circ F^{-1})}{\partial w_l \partial w_m} \frac{\partial f_l}{\partial z_j} \frac{\partial f_m}{\partial z_k} + \sum_m \frac{\partial (r \circ F^{-1})}{\partial w_m} \frac{\partial^2 f_m}{\partial z_j \partial z_k} \end{pmatrix}.$$

Here, a straightforward calculation shows that for a Möbius transformation,

$$\begin{aligned} \frac{\partial^2 f_m}{\partial z_j \partial z_k} &= -a_{m,j} \frac{a_{n+1,k}}{g_{n+1}^2} - a_{m,k} \frac{a_{n+1,j}}{g_{n+1}^2} + 2 g_m \frac{a_{n+1,j} a_{n+1,k}}{g_{n+1}^3} \\ &= -\frac{\partial f_m}{\partial z_j} \frac{a_{n+1,k}}{g_{n+1}} - \frac{\partial f_m}{\partial z_k} \frac{a_{n+1,j}}{g_{n+1}}. \end{aligned}$$

We can then perform row and column operations in order to simplify the matrix on the right-hand side of (5). In particular, we multiply the first column by $a_{n+1,k}/g_{n+1}$ and add to the $(k+1)$-st column; we also multiply the first row by $a_{n+1,j}/g_{n+1}$ and add to the $(j+1)$-st row. After doing this for all j, k, the sum that contains $\partial^2 f_m / \partial z_j \partial z_k$ has gone, so that taking the determinant of both sides of (5) proves identity (4), just like for the previous situation. □

From Proposition 1, it follows that if M is Levi non-degenerate and F is a Möbius transformation, then

$$(6) \qquad \frac{\mathcal{Q}_{r,p}}{\mathcal{L}_{r,p}} = \frac{\mathcal{Q}_{r \circ F^{-1}, F(p)}}{\mathcal{L}_{r \circ F^{-1}, F(p)}} \frac{\det F'(p)}{\overline{\det F'(p)}}.$$

In particular, for a fixed constant ε, the condition $\mathcal{Q}_{r,p} = \varepsilon \, \mathcal{L}_{r,p}$ is preserved by those F for which $\det F'$ is real. These are the affine maps described in Theorem 1. Meanwhile, the condition $\mathcal{Q}_{r,p} = \varepsilon \, \mathcal{L}_{r,p}$ for some constant ε is preserved by those F for which $\det F'$ is constant. These are the general affine maps of $\mathbb{C}^n$.

From (6), it is also a simple matter to reduce the proof of Theorem 1 to the case $\varepsilon \in \mathbb{R}^+ \setminus \{1\}$. In particular, if $M^3 \subset \mathbb{C}^2$ satisfies $\mathcal{Q}_{r,p} = \varepsilon \, \mathcal{L}_{r,p}$ for $\varepsilon \in \mathbb{C}$

with $|\varepsilon| \neq 0, 1$, then the affine transformation $F(z_1, z_2) = (z_1, e^{i(\arg \varepsilon)/2} z_2)$ results in a surface $F(M)$ for which $\mathcal{Q}_{r\circ F^{-1}, F(p)} = |\varepsilon| \, \mathcal{L}_{r\circ F^{-1}, F(p)}$. If Theorem 1 holds for $\varepsilon \in \mathbb{R}^+ \setminus \{1\}$, it follows that $F(M)$ is contained in the image of $M_{|\varepsilon|}$ under an affine map $G(w) = Aw + b$ where $0 \neq \det A \in \mathbb{R}$. Applying F^{-1}, it then follows that the original surface M is contained in $(F^{-1} \circ G)(M_{|\varepsilon|})$. Since $M_{|\varepsilon|} = F(M_\varepsilon)$, it follows that M is contained in the image of M_ε under the affine map $\widetilde{G} = F^{-1} \circ G \circ F$. If $\widetilde{G}$ is expressed as $\widetilde{G}(z) = \widetilde{A} z + \widetilde{b}$, then clearly $\det \widetilde{A} = \det A$, so that $0 \neq \det \widetilde{A} \in \mathbb{R}$, and the reduction is complete.

To conclude this section, we verify that condition (1) is independent of the choice of defining function. We return to the general case $M^{2n-1} \subset \mathbb{C}^n$.

Proposition 2. *Let r and $\widetilde{r}$ be defining functions for a twice differentiable hypersurface $M^{2n-1} \subset \mathbb{C}^n$ with $\widetilde{r} = h \cdot r$ for a twice differentiable function $h > 0$. Then on M, both $\mathcal{L}_{\widetilde{r},p} = h^{n+1} \mathcal{L}_{r,p}$ and $\mathcal{Q}_{\widetilde{r},p} = h^{n+1} \mathcal{Q}_{r,p}$. In particular, the quotient $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p}$ is independent of the choice of defining function.*

Proof. We establish $\mathcal{Q}_{\widetilde{r},p} = h^{n+1} \mathcal{Q}_{r,p}$. First, notice that

$$\frac{\partial(hr)}{\partial z_j} = \frac{\partial h}{\partial z_j} r + h \frac{\partial r}{\partial z_j}$$

and

$$\frac{\partial^2 (hr)}{\partial z_j \partial z_k} = \frac{\partial^2 h}{\partial z_j \partial z_k} r + \frac{\partial h}{\partial z_j} \frac{\partial r}{\partial z_k} + \frac{\partial h}{\partial z_k} \frac{\partial r}{\partial z_j} + h \frac{\partial^2 r}{\partial z_j \partial z_k}.$$

Then using $r(p) = 0$, as well as row and column operations similar to those in the second half of the proof of Proposition 1, it follows that

$$\mathcal{Q}_{h\cdot r,p} = -\det \begin{pmatrix} h\, r & h \dfrac{\partial r}{\partial z_k} \\ h \dfrac{\partial r}{\partial z_j} & h \dfrac{\partial^2 r}{\partial z_j \partial z_k} \end{pmatrix} = h^{n+1} \mathcal{Q}_{r,p}.$$

The identity $\mathcal{L}_{\widetilde{r},p} = h^{n+1} \mathcal{L}_{r,p}$ is handled similarly. □

3. Geometric structure of the quadratic forms

The proof of Theorem 1 uses classical differential geometry. We use the following notation, much of which can be found in Helgason [7] or Hicks [9]. For the time being, we continue to consider the case of general dimension. In the next section we restrict to the case $n = 2$.

Coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n$ correspond with coordinates $(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$ according to $z_j = x_j + i y_j$. Under this identification, the real Euclidean space inherits a complex structure $J : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$ that corresponds with multiplication by $i = \sqrt{-1}$ and is given by $J(\partial_{x_j}) = \partial_{y_j}$, $J(\partial_{y_j}) = -\partial_{x_j}$. This structure preserves the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $T\mathbb{R}^{2n}$. In fact, $J^* = -J$ and $J^2 = -I$. For

$X \in T\mathbb{R}^{2n}$, we let $\overline{d} = \overline{d}_X$ denote the standard (flat) connection on $\mathbb{R}^{2n}$. The complex structure and the connection commute with one another.

The real tangent space of $M = M^{2n-1}$ is denoted by TM. The complex tangent space is the codimension one subspace $HM = TM \cap J(TM)$. If M has defining function r, then a vector $X \in H_pM$ can be represented in coordinates by $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$ where $\sum r_j(p)s_j = 0$. The subscripts on r refer to holomorphic partial derivatives.

Let N be a unit normal vector on M. Then the direction orthogonal to HM in TM is JN. For $X \in TM$, let $d = d_X$ be the Riemannian connection that M inherits as a submanifold of $\mathbb{R}^n$. (It is exactly the restriction of $\overline{d} = \overline{d}_X$ to M.) Like $\overline{d}$, the connection is symmetric and metric, so $[X, Y] = d_XY - d_YX$ for $X, Y \in TM$, and $X\langle Y, Z\rangle = \langle d_XY, Z\rangle + \langle Y, d_XZ\rangle$ for $X, Y, Z \in TM$.

The Weingarten map is the operator $S : TM \to TM$ given by $S(X) = \overline{d}_XN$. This operator is self-adjoint. Related to S is the second fundamental form. This is the symmetric bilinear form $b(X, Y) = \langle S(X), Y\rangle = \langle \overline{d}_XN, Y\rangle$. The main structural equation for a hypersurface in Euclidean space is the Codazzi equation. It says that if $X, Y \in TM$, then

$$d_XS(Y) - d_YS(X) - S([X, Y]) = 0.$$

This vector equation describes the compatibility conditions between the induced metric and the second fundamental form for a hypersurface in Euclidean space.

The following proposition describes the geometric structure of the forms $L_{r,p}$ and $Q_{r,p}$. The expression for the Levi form was proved by Robert Hermann [8].

Proposition 3. *Let $M^{2n-1} \subset \mathbb{C}^n$ be a twice differentiable hypersurface and let r be a defining function for M normalized so that $|\nabla r| \equiv 2$ on M. Let $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$ be coordinates for $X \in H_pM$. Then,*

$$L_{r,p}(s, \overline{s}) = \frac{1}{2}\left(b(X, X) + b(JX, JX)\right)$$
$$Q_{r,p}(s, s) = \frac{1}{2}\left(b(X, X) - b(JX, JX)\right) - \frac{i}{2}\left(b(X, JX) + b(JX, X)\right)$$

Proof. The defining function has been normalized so that in coordinates, $N = (r_{\overline{1}}, \ldots, r_{\overline{n}})$. The subscripts refer to antiholomorphic partial derivatives; the factor of 2 compensates for the factor of 1/2 in $\partial_{\overline{z}_j} = (1/2)(\partial_{x_j} + i\partial_{y_j})$.

If $X = (s_1, \dots, s_n) \in H_pM$, then $JX = (is_1, \dots, is_n)$, and using the dot to represent the complex dot product, we find

$$\begin{aligned} b(X,X) &= \operatorname{Re}[\overline{d}_X N \cdot \overline{X}] \\ &= \operatorname{Re}\left(\textstyle\sum_{j=1}^n (s_j \partial_{z_j} + \overline{s}_j \partial_{\overline{z}_j})(r_{\overline{1}}, \dots, r_{\overline{n}}) \cdot (\overline{s}_1, \dots, \overline{s}_n)\right) \\ &= \operatorname{Re}\left(\textstyle\sum_{j,k=1}^n r_{j\overline{k}} s_j \overline{s}_k + r_{\overline{jk}} \overline{s}_j \overline{s}_k\right) = L_{r,p}(s, \overline{s}) + \operatorname{Re} Q_{r,p}(s,s), \end{aligned}$$

$$\begin{aligned} b(JX,JX) &= \operatorname{Re}[\overline{d}_{JX} N \cdot \overline{JX}] \\ &= \operatorname{Re}\left(\textstyle\sum_{j=1}^n (is_j \partial_{z_j} - i\overline{s}_j \partial_{\overline{z}_j})(r_{\overline{1}}, \dots, r_{\overline{n}}) \cdot (-i\overline{s}_1, \dots, -i\overline{s}_n)\right) \\ &= \operatorname{Re}\left(\textstyle\sum_{j,k=1}^n r_{j\overline{k}} s_j \overline{s}_k - r_{\overline{jk}} \overline{s}_j \overline{s}_k\right) = L_{r,p}(s, \overline{s}) - \operatorname{Re} Q_{r,p}(s,s), \end{aligned}$$

$$\begin{aligned} b(X,JX) = b(JX,X) &= \operatorname{Re}[\overline{d}_X N \cdot \overline{JX}] \\ &= \operatorname{Re}\left(\textstyle\sum_{j=1}^n (s_j \partial_{z_j} + \overline{s}_j \partial_{\overline{z}_j})(r_{\overline{1}}, \dots, r_{\overline{n}}) \cdot (-i\overline{s}_1, \dots, -i\overline{s}_n)\right) \\ &= \operatorname{Re}\left(\textstyle\sum_{j,k=1}^n -ir_{j\overline{k}} s_j \overline{s}_k - ir_{\overline{jk}} \overline{s}_j \overline{s}_k\right) = -\operatorname{Im} Q_{r,p}(s,s). \end{aligned}$$

The expressions for $L_{r,p}(s,\overline{s})$ and $Q_{r,p}(s,s)$ follow directly from these calculations.

□

4. Proof of Theorem 1

The proof of Theorem 1 is similar to the proof the author used to prove Theorem 2. It makes extensive use of the structural equations for a hypersurface. The strategy is to identify a vanishing quantity on M, then to use it to identify constant directions in $\mathbb{C}^2$ as observed from M. Following a suitable affine transformation, the cross-sections of M are ellipses or hyperbolas. With this extra restriction on M, and after a further normalization, it is shown that condition (1) requires that M is contained in a surface M_ε.

We restrict to the case $n = 2$. Let r be a defining function that is normalized so that $|\nabla r| \equiv 2$. Then condition (1) can be rewritten

$$Q_{r,p}((-r_2, r_1), (-r_2, r_1)) = \varepsilon L_{r,p}((-r_2, r_1), (-r_{\overline{2}}, r_{\overline{1}})). \tag{7}$$

Using the remark that follows Proposition 1 we assume that $\varepsilon \in \mathbb{R}^+ \setminus \{1\}$. From now on we also use the preferred orthonormal system,

$$N = (r_{\overline{1}}, r_{\overline{2}}), \quad JN = (ir_{\overline{1}}, ir_{\overline{2}}), \quad X = (-r_2, r_1), \quad JX = (-ir_2, ir_1). \tag{8}$$

By Proposition 3 it follows that $b(X,JX) = b(JX,X) = 0$, and

$$b(X,X) - b(JX,JX) = \varepsilon\, (b(X,X) + b(JX,JX)).$$

In particular, $b(X,X) = \lambda(1+\varepsilon)$ and $b(JX,JX) = \lambda(1-\varepsilon)$ where λ is real and $\lambda \not\equiv 0$.

The second fundamental form for $M^3 \subset \mathbb{C}^2$ can then be represented by the 3×3 matrix of real functions

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \lambda(1+\varepsilon) & 0 \\ \gamma & 0 & \lambda(1-\varepsilon) \end{pmatrix}.$$

The rows and columns of the matrix correspond with the tangent vectors JN, X, JX, respectively, as defined in (8). Our proof will be local, so using the smoothness of M we assume $\lambda \neq 0$. The Weingarten map can be read from the second fundamental form,

$$\begin{aligned} S(JN) &= \alpha JN + \beta X + \gamma JX, \\ S(X) &= \beta JN + \lambda(1+\varepsilon)X, \\ S(JX) &= \gamma JN + \lambda(1-\varepsilon)JX. \end{aligned}$$

Our first step shows how the system (8) is useful for computing the connection along M.

Lemma 1. *Let $M^3 \subset \mathbb{C}^2$ be twice differentiable and have second fundamental form as described above. If $Y \in TM$ then $\langle \overline{d}_Y X, JX\rangle = -\langle JN, \overline{d}_Y N\rangle$. In particular,*

$$\begin{aligned} \langle \overline{d}_{JN} X, JX\rangle &= -\alpha, \\ \langle \overline{d}_{X} X, JX\rangle &= -\beta, \\ \langle \overline{d}_{JX} X, JX\rangle &= -\gamma. \end{aligned}$$

Proof. Using the dot to represent the complex dot product, we find

$$\begin{aligned} \langle \overline{d}_Y X, JX\rangle = \mathrm{Re}\,[Y(X)\cdot \overline{JX}] &= \mathrm{Re}\,[Y(-r_2, r_1)\cdot (ir_{\overline{2}}, -ir_{\overline{1}})] \\ &= -\mathrm{Re}\,[Y(r_2, r_1)\cdot (ir_{\overline{2}}, ir_{\overline{1}})] \\ &= -\mathrm{Re}\,[Y(r_1, r_2)\cdot (ir_{\overline{1}}, ir_{\overline{2}})] \\ &= -\mathrm{Re}\,[(ir_{\overline{1}}, ir_{\overline{2}})\cdot Y(r_1, r_2)] \\ &= -\mathrm{Re}\,[JN\cdot \overline{Y(N)}] \\ &= -\langle JN, \overline{d}_Y N\rangle. \end{aligned}$$

The remaining claims are special cases of this fact. □

The connection along M is described nicely using entries from the second fundamental form.

Lemma 2. *Let $M^3 \subset \mathbb{C}^2$ be twice differentiable and have second fundamental form as described above. Then the connection on $\mathbb{C}^2$ along M is given by*

$$\overline{d}_{JN}N = +\alpha JN + \beta X + \gamma JX, \tag{9}$$

$$\overline{d}_{JN}JN = -\alpha N - \gamma X + \beta JX, \tag{10}$$

$$\overline{d}_{JN}X = -\beta N + \gamma JN - \alpha JX, \tag{11}$$

$$\overline{d}_{JN}JX = -\gamma N - \beta JN + \alpha X, \tag{12}$$

$$\overline{d}_{X}N = +\beta JN + \lambda(1+\varepsilon)X, \tag{13}$$

$$\overline{d}_{X}JN = -\beta N + \lambda(1+\varepsilon)JX, \tag{14}$$

$$\overline{d}_{X}X = -\lambda(1+\varepsilon)N - \beta JX, \tag{15}$$

$$\overline{d}_{X}JX = -\lambda(1+\varepsilon)JN + \beta X, \tag{16}$$

$$\overline{d}_{JX}N = +\gamma JN + \lambda(1-\varepsilon)JX, \tag{17}$$

$$\overline{d}_{JX}JN = -\gamma N - \lambda(1-\varepsilon)X, \tag{18}$$

$$\overline{d}_{JX}X = +\lambda(1-\varepsilon)JN - \gamma JX, \tag{19}$$

$$\overline{d}_{JX}JX = -\lambda(1-\varepsilon)N + \gamma X. \tag{20}$$

Proof. Identities (9), (13), and (17) can be read directly from the second fundamental form, because if $Y \in TM$, then $\langle \overline{d}_Y N, N\rangle = (1/2)Y(\langle N, N\rangle) = 0$. We also give proofs for (10) and (11). First,

$$\begin{aligned}
\langle \overline{d}_{JN}JN, N\rangle &= -\langle JN, \overline{d}_{JN}N\rangle = -\alpha,\\
\langle \overline{d}_{JN}JN, JN\rangle &= (1/2)JN(\langle JN, JN\rangle) = 0,\\
\langle \overline{d}_{JN}JN, X\rangle &= \langle J\overline{d}_{JN}N, X\rangle \;=\; -\langle \overline{d}_{JN}N, JX\rangle = -\gamma,\\
\langle \overline{d}_{JN}JN, JX\rangle &= \langle J\overline{d}_{JN}N, JX\rangle \;=\; \langle \overline{d}_{JN}N, X\rangle = \beta.
\end{aligned}$$

Together, these computations prove that $\overline{d}_{JN}JN = -\alpha N - \gamma X + \beta JX$. Similarly,

$$\begin{aligned}
\langle \overline{d}_{JN}X, N\rangle &= -\langle X, \overline{d}_{JN}N\rangle = -\beta,\\
\langle \overline{d}_{JN}X, JN\rangle &= -\langle X, \overline{d}_{JN}JN\rangle = -\langle X, J\overline{d}_{JN}N\rangle = \langle JX, \overline{d}_{JN}N\rangle = \gamma,\\
\langle \overline{d}_{JN}X, X\rangle &= (1/2)JN(\langle X, X\rangle) = 0,\\
\langle \overline{d}_{JN}X, JX\rangle &= -\alpha.
\end{aligned}$$

(The last identity uses Lemma 1.) Together, these prove that $\overline{d}_{JN}X = -\beta N + \gamma JN - \alpha JX$. The remaining identities use similar reasoning. □

It is then a simple matter to describe the connection that M inherits as a submanifold of $\mathbb{C}^2$.

Lemma 3. *Let $M^3 \subset \mathbb{C}^2$ be twice differentiable and have second fundamental form as described above. Then the connection on M is given by*

$$d_{JN}JN = -\gamma X + \beta JX, \tag{21}$$
$$d_{JN}X = +\gamma JN - \alpha JX, \tag{22}$$
$$d_{JN}JX = -\beta JN + \alpha X, \tag{23}$$
$$d_X JN = +\lambda(1+\varepsilon)JX, \tag{24}$$
$$d_X X = -\beta JX, \tag{25}$$
$$d_X JX = -\lambda(1+\varepsilon)JN + \beta X, \tag{26}$$
$$d_{JX}JN = -\lambda(1-\varepsilon)X, \tag{27}$$
$$d_{JX}X = +\lambda(1-\varepsilon)JN - \gamma JX, \tag{28}$$
$$d_{JX}JX = +\gamma X. \tag{29}$$

Proof. These identities follow immediately from Lemma 2. One ignores the normal components and retains the tangential components. □

The Codazzi equation reveals several restrictions on the second fundamental form.

Lemma 4. *Suppose $M^3 \subset \mathbb{C}^2$ is three times differentiable and has second fundamental form as described above. If $\lambda \neq 0$, then*

$$JX(\lambda) = +3\beta\lambda\frac{1-\varepsilon}{1+\varepsilon}, \tag{30}$$
$$X(\lambda) = -3\gamma\lambda\frac{1+\varepsilon}{1-\varepsilon}, \tag{31}$$
$$JX(\alpha) = JN(\gamma) - 3\beta\lambda(1-\varepsilon), \tag{32}$$
$$X(\alpha) = JN(\beta) + 3\gamma\lambda(1+\varepsilon), \tag{33}$$
$$JX(\beta) = -2\gamma^2 + \beta^2 - \lambda^2(1-\varepsilon^2) + \alpha\lambda(1-3\varepsilon), \tag{34}$$
$$X(\beta) = (1+\varepsilon)JN(\lambda) - 3\beta\gamma, \tag{35}$$
$$JX(\gamma) = (1-\varepsilon)JN(\lambda) + 3\beta\gamma, \tag{36}$$
$$X(\gamma) = +2\beta^2 - \gamma^2 + \lambda^2(1-\varepsilon^2) - \alpha\lambda(1+3\varepsilon). \tag{37}$$

Proof. We simply apply the Codazzi equation to the different pairs of tangent vectors.

(i) Applying the equation to X, JX,

$$\begin{aligned}
0 &= d_X S(JX) - d_{JX} S(X) - S(d_X JX - d_{JX} X) \\
&= d_X(\gamma JN + \lambda(1-\varepsilon)JX) - d_{JX}(\beta JN + \lambda(1+\varepsilon)X) + S(2\lambda JN - \beta X - \gamma JX) \\
&= X(\gamma)JN + \gamma d_X JN + X(\lambda)(1-\varepsilon)JX + \lambda(1-\varepsilon) d_X JX \\
&\quad - JX(\beta)JN - \beta d_{JX} JN - JX(\lambda)(1+\varepsilon)X - \lambda(1+\varepsilon) d_{JX} X \\
&\quad + 2\lambda(\alpha JN + \beta X + \gamma JX) - \beta(\beta JN + \lambda(1+\varepsilon)X) - \gamma(\gamma JN + \lambda(1-\varepsilon)JX) \\
&= a_1 JN + a_2 X + a_3 JX,
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= X(\gamma) - JX(\beta) - 2\lambda^2(1-\varepsilon^2) + 2\alpha\lambda - \beta^2 - \gamma^2 \\
a_2 &= -JX(\lambda)(1+\varepsilon) + 3\beta\lambda(1-\varepsilon) \\
a_3 &= X(\lambda)(1-\varepsilon) + 3\gamma\lambda(1+\varepsilon).
\end{aligned}$$

Equations (30) and (31) follow from the requirement that $a_2 = 0$ and $a_3 = 0$. (The vanishing of a_1 is reflected by (34) and (37); these are proved in what follows.)

(ii) Applying the equation to JX, JN,

$$\begin{aligned}
0 &= d_{JX} S(JN) - d_{JN} S(JX) - S(d_{JX} JN - d_{JN} JX) \\
&= d_{JX}(\alpha JN + \beta X + \gamma JX) - d_{JN}(\gamma JN + \lambda(1-\varepsilon)JX) \\
&\quad - S(\beta JN - (\alpha + \lambda(1-\varepsilon))X) \\
&= JX(\alpha)JN + \alpha d_{JX} JN + JX(\beta)X + \beta d_{JX} X + JX(\gamma)JX + \gamma d_{JX} JX \\
&\quad - JN(\gamma)JN - \gamma d_{JN} JN - JN(\lambda)(1-\varepsilon)JX - \lambda(1-\varepsilon) d_{JN} JX \\
&\quad - \beta(\alpha JN + \beta X + \gamma JX) + (\alpha + \lambda(1-\varepsilon))(\beta JN + \lambda(1+\varepsilon)X) \\
&= a_1 JN + a_2 X + a_3 JX,
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= JX(\alpha) - JN(\gamma) + 3\beta\lambda(1-\varepsilon) \\
a_2 &= JX(\beta) + 2\gamma^2 - \beta^2 + \lambda^2(1-\varepsilon^2) - \alpha\lambda(1-3\varepsilon) \\
a_3 &= JX(\gamma) - (1-\varepsilon)JN(\lambda) - 3\beta\gamma.
\end{aligned}$$

Equations (32), (34), and (36) follow from the requirement that $a_1 = 0$, $a_2 = 0$, and $a_3 = 0$.

(iii) Applying the equation to X, JN,

$$\begin{aligned}
0 &= d_X S(JN) - d_{JN} S(X) - S(d_X JN - d_{JN} X) \\
&= d_X(\alpha JN + \beta X + \gamma JX) - d_{JN}(\beta JN + \lambda(1+\varepsilon)X) \\
&\quad + S(\gamma JN - (\alpha + \lambda(1+\varepsilon))JX) \\
&= X(\alpha)JN + \alpha d_X JN + X(\beta)X + \beta d_X X + X(\gamma)JX + \gamma d_X JX \\
&\quad - JN(\beta)JN - \beta d_{JN} JN - JN(\lambda)(1+\varepsilon)X - \lambda(1+\varepsilon) d_{JN} X \\
&\quad + \gamma(\alpha JN + \beta X + \gamma JX) - (\alpha + \lambda(1+\varepsilon))(\gamma JN + \lambda(1-\varepsilon)JX) \\
&= a_1 JN + a_2 X + a_3 JX,
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= X(\alpha) - JN(\beta) - 3\gamma\lambda(1+\varepsilon) \\
a_2 &= X(\beta) - (1+\varepsilon)JN(\lambda) + 3\beta\gamma \\
a_3 &= X(\gamma) - 2\beta^2 + \gamma^2 - \lambda^2(1-\varepsilon^2) + \alpha\lambda(1+3\varepsilon).
\end{aligned}$$

Equations (33), (35), and (37) follow from the requirement that $a_1 = 0$, $a_2 = 0$, and $a_3 = 0$. □

The symmetry of the connection leads to further restrictions.

Lemma 5. *Let $M^3 \subset \mathbb{C}^2$ be as described above. If $\lambda \neq 0$, then*

$$JN(\lambda) = -6\beta\gamma \frac{\varepsilon}{1-\varepsilon^2}, \tag{38}$$

$$X(\beta) = -3\beta\gamma \frac{1+\varepsilon}{1-\varepsilon}, \tag{39}$$

$$JX(\gamma) = +3\beta\gamma \frac{1-\varepsilon}{1+\varepsilon}, \tag{40}$$

$$JN(\beta) = \frac{\gamma^3}{\lambda} \frac{4\varepsilon}{(1-\varepsilon)^2} - \alpha\gamma \frac{1+5\varepsilon}{1-\varepsilon} - \gamma\lambda(1+\varepsilon), \tag{41}$$

$$JN(\gamma) = \frac{\beta^3}{\lambda} \frac{4\varepsilon}{(1+\varepsilon)^2} + \alpha\beta \frac{1-5\varepsilon}{1+\varepsilon} + \beta\lambda(1-\varepsilon), \tag{42}$$

$$X(\alpha) = \frac{\gamma^3}{\lambda} \frac{4\varepsilon}{(1-\varepsilon)^2} - \alpha\gamma \frac{1+5\varepsilon}{1-\varepsilon} + 2\gamma\lambda(1+\varepsilon), \tag{43}$$

$$JX(\alpha) = \frac{\beta^3}{\lambda} \frac{4\varepsilon}{(1+\varepsilon)^2} + \alpha\beta \frac{1-5\varepsilon}{1+\varepsilon} - 2\beta\lambda(1-\varepsilon). \tag{44}$$

Proof. We apply the identity $[X, JX] = d_X JX - d_{JX} X$ to each of λ, β, and γ. Using Lemma 3, this identity can be rewritten $[X, JX] = -2\lambda JN + \beta X + \gamma JX$. We also make frequent use of the identities proved in Lemma 4.

(i) Using (30) and (31), and then (35) and (36), we find

$$\begin{aligned}[X,JX](\lambda) =& X(3\beta\lambda\frac{1-\varepsilon}{1+\varepsilon}) - JX(-3\gamma\lambda\frac{1+\varepsilon}{1-\varepsilon})\\ =& 3\left((1+\varepsilon)JN(\lambda) - 3\beta\gamma\right)\lambda\frac{1-\varepsilon}{1+\varepsilon} + 3\beta X(\lambda)\frac{1-\varepsilon}{1+\varepsilon}\\ &+ 3\left((1-\varepsilon)JN(\lambda) + 3\beta\gamma\right)\lambda\frac{1+\varepsilon}{1-\varepsilon} + 3\gamma JX(\lambda)\frac{1+\varepsilon}{1-\varepsilon}.\end{aligned}$$

So the identity $[X,JX](\lambda) = (-2\lambda JN + \beta X + \gamma JX)(\lambda)$ can be rewritten

$$8\lambda JN(\lambda) = \beta X(\lambda)(1-3\frac{1-\varepsilon}{1+\varepsilon}) + \gamma JX(\lambda)(1-3\frac{1+\varepsilon}{1-\varepsilon}) + 9\beta\gamma\lambda(\frac{1-\varepsilon}{1+\varepsilon} - \frac{1+\varepsilon}{1-\varepsilon}).$$

Using (30) and (31), and simplifying, proves (38) since $\lambda \neq 0$. Then again using (35) and (36), it proves (39) and (40).

(ii) Using (34) and (39), and then (33), we find

$$\begin{aligned}[X,JX](\beta) =& X(-2\gamma^2 + \beta^2 - \lambda^2(1-\varepsilon^2) + \alpha\lambda(1-3\varepsilon)) - JX(-3\beta\gamma\frac{1+\varepsilon}{1-\varepsilon})\\ =& -4\gamma X(\gamma) + 2\beta X(\beta) - 2\lambda(1-\varepsilon^2)X(\lambda)\\ &+ \left(JN(\beta) + 3\gamma\lambda(1+\varepsilon)\right)\lambda(1-3\varepsilon) + \alpha X(\lambda)(1-3\varepsilon)\\ &+ 3\gamma JX(\beta)\frac{1+\varepsilon}{1-\varepsilon} + 3\beta JX(\gamma)\frac{1+\varepsilon}{1-\varepsilon}.\end{aligned}$$

So the identity $[X,JX](\beta) = (-2\lambda JN + \beta X + \gamma JX)(\beta)$ can be rewritten

$$\begin{aligned}3\lambda JN(\beta)(1-\varepsilon) =& 4\gamma X(\gamma) - \beta X(\beta) + 2\lambda(1-\varepsilon^2)X(\lambda) - 3\gamma\lambda^2(1+\varepsilon)(1-3\varepsilon)\\ &- \alpha X(\lambda)(1-3\varepsilon) + \gamma JX(\beta)(1-3\frac{1+\varepsilon}{1-\varepsilon}) - 3\beta JX(\gamma)\frac{1+\varepsilon}{1-\varepsilon}.\end{aligned}$$

Using (31), (34), (37), (39) and (40), and simplifying, proves (41) since $\lambda \neq 0$ and $\varepsilon \neq 1$. Then again using (33), it proves (43).

(iii) Using (37) and (40), and then (32), we find

$$\begin{aligned}[X,JX](\gamma) =& X(3\beta\gamma\,\frac{1-\varepsilon}{1+\varepsilon}) - JX(2\beta^2 - \gamma^2 + \lambda^2(1-\varepsilon^2) - \alpha\lambda(1+3\varepsilon))\\ =& 3\gamma X(\beta)\frac{1-\varepsilon}{1+\varepsilon} + 3\beta X(\gamma)\frac{1-\varepsilon}{1+\varepsilon}\\ &- 4\beta JX(\beta) + 2\gamma JX(\gamma) - 2\lambda(1-\varepsilon^2)JX(\lambda)\\ &+ \left(JN(\gamma) - 3\beta\lambda(1-\varepsilon)\right)\lambda(1+3\varepsilon) + \alpha JX(\lambda)(1+3\varepsilon).\end{aligned}$$

So the identity $[X,JX](\gamma) = (-2\lambda JN + \beta X + \gamma JX)(\gamma)$ can be rewritten

$$\begin{aligned}3\lambda JN(\gamma)(1+\varepsilon) =& -3\gamma X(\beta)\frac{1-\varepsilon}{1+\varepsilon} + \beta X(\gamma)(1-3\frac{1-\varepsilon}{1+\varepsilon}) + 4\beta JX(\beta) - \gamma JX(\gamma)\\ &+ 2\lambda(1-\varepsilon^2)JX(\lambda) + 3\beta\lambda^2(1-\varepsilon)(1+3\varepsilon) - \alpha JX(\lambda)(1+3\varepsilon).\end{aligned}$$

Using (30), (34), (37), (39) and (40), and simplifying, proves (42) since $\lambda \neq 0$ and $\varepsilon \neq -1$. Then again using (32), it proves (44). $\square$

Next, the symmetry of the connection can be used to identify a function that vanishes.

Lemma 6. *Let $M^3 \subset \mathbb{C}^2$ be as described above. If $\lambda \neq 0$, then*

$$\frac{\beta^2}{1+\varepsilon} + \frac{\gamma^2}{1-\varepsilon} - \alpha\lambda = 0.$$

Proof. In particular, we apply the identity $[X, JN] = d_X JN - d_{JN} X$ to β. By Lemma 3, we have $[X, JN] = -\gamma JN + (\alpha + \lambda(1+\varepsilon))JX$.

Then using (39) and (41), it follows that

$$\begin{aligned}
0 &= [X, JN](\beta) - (d_X JN - d_{JN} X)(\beta) \\
&= X\left(\frac{\gamma^3}{\lambda}\frac{4\varepsilon}{(1-\varepsilon)^2} - \alpha\gamma\frac{1+5\varepsilon}{1-\varepsilon} - \gamma\lambda(1+\varepsilon)\right) - JN\left(-3\beta\gamma\frac{1+\varepsilon}{1-\varepsilon}\right) \\
&\quad + \gamma JN(\beta) - (\alpha + \lambda(1+\varepsilon))JX(\beta) \\
&= X(\gamma)\left(\frac{3\gamma^2}{\lambda}\frac{4\varepsilon}{(1-\varepsilon)^2} - \alpha\frac{1+5\varepsilon}{1-\varepsilon} - \lambda(1+\varepsilon)\right) - X(\alpha)\left(\gamma\frac{1+5\varepsilon}{1-\varepsilon}\right) \\
&\quad + X(\lambda)\left(-\frac{\gamma^3}{\lambda^2}\frac{4\varepsilon}{(1-\varepsilon)^2} - \gamma(1+\varepsilon)\right) \\
&\quad + JN(\beta)\left(3\gamma\frac{1+\varepsilon}{1-\varepsilon} + \gamma\right) + JN(\gamma)\left(3\beta\frac{1+\varepsilon}{1-\varepsilon}\right) - JX(\beta)(\alpha + \lambda(1+\varepsilon)) \\
&= \frac{12\,\varepsilon\,(1+\varepsilon)}{\lambda\,(1-\varepsilon)}\left(\frac{\beta^2}{1+\varepsilon} + \frac{\gamma^2}{1-\varepsilon} - \alpha\lambda\right)^2.
\end{aligned}$$

The last step uses (31), (34), (37), (41), (42) and (43), and a good deal of algebra. (The lengthy details are omitted.) Since $\lambda \neq 0$ and $\varepsilon \neq 0, \pm 1$, the lemma is proved. ☐

Finally, it is possible to identify a set of constant ambient directions (in $\mathbb{C}^2$).

Lemma 7. *Defined on M, the vectors*

$$Y \stackrel{\text{def}}{=} \lambda^{-2/3}\left(\lambda X + \frac{\gamma}{1-\varepsilon}N + \frac{\beta}{1+\varepsilon}JN\right), \tag{45}$$

$$JY = \lambda^{-2/3}\left(\lambda JX - \frac{\beta}{1+\varepsilon}N + \frac{\gamma}{1-\varepsilon}JN\right), \tag{46}$$

$$Z \stackrel{\text{def}}{=} \lambda^{-2/3}\left(\lambda N - \frac{\gamma}{1-\varepsilon}X + \frac{\beta}{1+\varepsilon}JX\right), \tag{47}$$

$$JZ = \lambda^{-2/3}\left(\lambda JN - \frac{\beta}{1+\varepsilon}X - \frac{\gamma}{1-\varepsilon}JX\right) \tag{48}$$

are constant.

Proof. To prove that Y is constant, we use the previous lemmas and show that each of the vectors $\overline{d}_{JN}Y$, $\overline{d}_X Y$, and $\overline{d}_{JX}Y$ is zero.

(i) Since

$$
\begin{aligned}
&\overline{d}_{JN}\left(\lambda X+\frac{\gamma}{1-\varepsilon}\ N+\frac{\beta}{1+\varepsilon}JN\right)\\
&\qquad =\lambda(-\beta N+\gamma JN-\alpha JX)+JN(\lambda)X+\frac{\gamma}{1-\varepsilon}(\alpha JN+\beta X+\gamma JX)\\
&\qquad\quad +\frac{JN(\gamma)}{1-\varepsilon}\,N+\frac{\beta}{1+\varepsilon}(-\alpha N-\gamma X+\beta JX)+\frac{JN(\beta)}{1+\varepsilon}JN
\end{aligned}
$$

and

$$
\lambda^{2/3}\,JN(\lambda^{-2/3})=-\frac{2}{3}\,\frac{JN(\lambda)}{\lambda}=\frac{\beta\gamma}{\lambda}\,\frac{4\varepsilon}{1-\varepsilon^2},
$$

it follows that

$$
\begin{aligned}
&\lambda^{2/3}\,\overline{d}_{JN}Y\\
&\quad =\frac{\beta\gamma}{\lambda}\,\frac{4\varepsilon}{1-\varepsilon^2}\left(\lambda X+\frac{\gamma}{1-\varepsilon}N+\frac{\beta}{1+\varepsilon}JN\right)+\overline{d}_{JN}\left(\lambda X+\frac{\gamma}{1-\varepsilon}N+\frac{\beta}{1+\varepsilon}JN\right)\\
&\quad =a_1X+a_2JX+a_3N+a_4JN,
\end{aligned}
$$

where

$$
\begin{aligned}
a_1&=4\beta\gamma\frac{\varepsilon}{1-\varepsilon^2}+JN(\lambda)+\beta\gamma\frac{1}{1-\varepsilon}-\beta\gamma\frac{1}{1+\varepsilon}\\
a_2&=-\alpha\lambda+\gamma^2\frac{1}{1-\varepsilon}+\beta^2\frac{1}{1+\varepsilon}\\
a_3&=\frac{\beta\gamma^2}{\lambda}\,\frac{4\varepsilon}{(1-\varepsilon^2)(1-\varepsilon)}-\beta\lambda+\frac{JN(\gamma)}{1-\varepsilon}-\alpha\beta\frac{1}{1+\varepsilon}\\
a_4&=\frac{\beta^2\gamma}{\lambda}\,\frac{4\varepsilon}{(1-\varepsilon^2)(1+\varepsilon)}+\gamma\lambda+\alpha\gamma\frac{1}{1-\varepsilon}+\frac{JN(\beta)}{1+\varepsilon}.
\end{aligned}
$$

Each of the coefficients a_j is zero, as follows from (38), (41), (42), and Lemma 6. Since $\lambda\neq 0$, it follows that $\overline{d}_{JN}Y=0$.

(ii) Since

$$
\begin{aligned}
&\overline{d}_{X}\left(\lambda X+\frac{\gamma}{1-\varepsilon}\ N+\frac{\beta}{1+\varepsilon}JN\right)\\
&\qquad =\lambda(-\lambda(1+\varepsilon)N-\beta JX)+X(\lambda)\,X+\frac{\gamma}{1-\varepsilon}\,(\beta JN+\lambda(1+\varepsilon)X)\\
&\qquad\quad +\frac{X(\gamma)}{1-\varepsilon}\,N+\frac{\beta}{1+\varepsilon}\,(-\beta N+\lambda(1+\varepsilon)JX)+\frac{X(\beta)}{1+\varepsilon}\,JN
\end{aligned}
$$

and

$$
\lambda^{2/3}\,X(\lambda^{-2/3})=-\frac{2}{3}\,\frac{X(\lambda)}{\lambda}=2\gamma\,\frac{1+\varepsilon}{1-\varepsilon},
$$

it follows that

$$
\begin{aligned}
&\lambda^{2/3}\,\overline{d}_X Y \\
&\quad = 2\gamma\,\frac{1+\varepsilon}{1-\varepsilon}\left(\lambda X+\frac{\gamma}{1-\varepsilon}N+\frac{\beta}{1+\varepsilon}JN\right)+\overline{d}_X\left(\lambda X+\frac{\gamma}{1-\varepsilon}N+\frac{\beta}{1+\varepsilon}JN\right)\\
&\quad = a_1X+a_2JX+a_3N+a_4JN,
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= 3\gamma\lambda\frac{1+\varepsilon}{1-\varepsilon}+X(\lambda)\\
a_2 &= 0\\
a_3 &= 2\gamma^2\frac{1+\varepsilon}{(1-\varepsilon)^2}-\lambda^2(1+\varepsilon)+\frac{X(\gamma)}{1-\varepsilon}-\beta^2\frac{1}{1+\varepsilon}\\
a_4 &= 3\beta\gamma\frac{1}{1-\varepsilon}+\frac{X(\beta)}{1+\varepsilon}.
\end{aligned}
$$

Each of the coefficients a_j is zero, as follows from (31), (37), and (39), and Lemma 6. Since $\lambda\neq 0$, it follows that $\overline{d}_X Y=0$.

(iii) Since

$$
\begin{aligned}
&\overline{d}_{JX}\left(\lambda X+\frac{\gamma}{1-\varepsilon}\,N+\frac{\beta}{1+\varepsilon}JN\right)\\
&\qquad = \lambda(\lambda(1-\varepsilon)JN-\gamma JX)+JX(\lambda)X+\frac{\gamma}{1-\varepsilon}(\gamma JN+\lambda(1-\varepsilon)JX)\\
&\qquad\quad +\frac{JX(\gamma)}{1-\varepsilon}N+\frac{\beta}{1+\varepsilon}(-\gamma N-\lambda(1-\varepsilon)X)+\frac{JX(\beta)}{1+\varepsilon}JN
\end{aligned}
$$

and

$$
\lambda^{2/3}\,JX(\lambda^{-2/3})=-\frac{2}{3}\,\frac{JX(\lambda)}{\lambda}=-2\beta\,\frac{1-\varepsilon}{1+\varepsilon},
$$

it follows that

$$
\begin{aligned}
&\lambda^{2/3}\,\overline{d}_{JX}Y\\
&\quad = -2\beta\,\frac{1-\varepsilon}{1+\varepsilon}\left(\lambda X+\frac{\gamma}{1-\varepsilon}N+\frac{\beta}{1+\varepsilon}JN\right)+\overline{d}_{JX}\left(\lambda X+\frac{\gamma}{1-\varepsilon}N+\frac{\beta}{1+\varepsilon}JN\right)\\
&\quad = a_1X+a_2JX+a_3N+a_4JN,
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= -3\beta\lambda\,\frac{1-\varepsilon}{1+\varepsilon}+JX(\lambda)\\
a_2 &= 0\\
a_3 &= -3\beta\gamma\,\frac{1}{1+\varepsilon}+\frac{JX(\gamma)}{1-\varepsilon}\\
a_4 &= -2\beta^2\,\frac{1-\varepsilon}{(1+\varepsilon)^2}+\lambda^2(1-\varepsilon)+\gamma^2\frac{1}{1-\varepsilon}+\frac{JX(\beta)}{1+\varepsilon}.
\end{aligned}
$$

Each of the coefficients a_j is zero, as follows from (30), (34), (40), and Lemma 6. Since $\lambda \neq 0$, it follows that $\overline{d}_{JX}Y = 0$.

We have therefore proved that Y is constant, and it follows that JY is constant as well. The proof for Z and JZ can be done in a similar fashion. Alternatively, after expressing all four vectors in terms of the defining function and using (8), one can see that Y being constant implies that Z and JZ are constant, too. □

Following Lemma 7, we apply a special unitary transformation (such a transformation is affine with real determinant) that orients the surface in $\mathbb{C}^2$ so that Z is parallel with $\partial/\partial x_1$. It then automatically follows that JZ is parallel with $\partial/\partial y_1$. In fact, Y and JY then also are parallel with $\partial/\partial x_2$ and $\partial/\partial y_2$, respectively. This can be seen by comparing the system of vectors in (8) with the definitions in Lemma 7.

Furthermore, since the four vectors in Lemma 7 are constant, their length, too, must be constant. So the positive quantity

$$\Lambda = \frac{1}{\lambda^{4/3}}\left(\frac{\beta^2}{(1+\varepsilon)^2} + \frac{\gamma^2}{(1-\varepsilon)^2} + \lambda^2\right)$$

is constant. We now apply a dilation that is uniform in all directions (and is therefore affine with real determinant), so that $\Lambda = 1$. This is possible because the curvatures vary inversely with the dilation factor, and Λ is homogeneous of degree $2/3$ with respect to the curvatures. So if on the initial surface $\Lambda = k$, then after a dilation by $k^{3/2}$, the new surface has $\Lambda = 1$. The normalization can also be written

$$\frac{\beta^2}{(1+\varepsilon)^2} + \frac{\gamma^2}{(1-\varepsilon)^2} = \lambda^{4/3} - \lambda^2 \tag{49}$$

Given now that the constant vectors are properly oriented and have unit length, we can say definitively that $Z = \partial/\partial x_1$, $JZ = \partial/\partial y_1$, $Y = \partial/\partial x_2$, and $JY = \partial/\partial y_2$. We proceed to show that M is invariant under translations in the $\partial/\partial y_1$ direction.

Lemma 8. *Let $M^3 \subset \mathbb{C}^2$ be as described above. If $\lambda \neq 0$, then M is JZ invariant. In particular, M can foliated by lines (or line segments) that are parallel with the y_1-axis.*

Proof. We show that all curvature information is unchanged by translations in the JZ direction. In particular, we will verify that $JZ(\beta) = 0$, $JZ(\gamma) = 0$, and $JZ(\lambda) = 0$. To prove $JZ(\beta) = 0$ we use (34), (39), and (41), and Lemma 6, and we find

$$\begin{aligned}
\lambda^{2/3} JZ(\beta) &= \lambda JN(\beta) - \frac{\beta}{1+\varepsilon} X(\beta) - \frac{\gamma}{1-\varepsilon} JX(\beta) \\
&= \lambda \left[\frac{\gamma^3}{\lambda} \frac{4\varepsilon}{(1-\varepsilon)^2} - \frac{\gamma}{\lambda} \left(\frac{\beta^2}{1+\varepsilon} + \frac{\gamma^2}{1-\varepsilon} \right) \frac{1+5\varepsilon}{1-\varepsilon} - \gamma\lambda(1+\varepsilon) \right] \\
&\quad - \frac{\beta}{1+\varepsilon} (-3\beta\gamma) \frac{1+\varepsilon}{1-\varepsilon} \\
&\quad - \frac{\gamma}{1-\varepsilon} \left[-2\gamma^2 + \beta^2 - \lambda^2(1-\varepsilon^2) + \left(\frac{\beta^2}{1+\varepsilon} + \frac{\gamma^2}{1-\varepsilon} \right) (1-3\varepsilon) \right] \\
&= 0.
\end{aligned}$$

(The simplification in the last step is best done by isolating the terms containing γ^3, $\beta^2\gamma$, and $\gamma\lambda^2$.) To prove $JZ(\gamma) = 0$ we use (37), (40), and (42), and Lemma 6, and we find

$$\begin{aligned}
\lambda^{2/3} JZ(\gamma) &= \lambda JN(\gamma) - \frac{\beta}{1+\varepsilon} X(\gamma) - \frac{\gamma}{1-\varepsilon} JX(\gamma) \\
&= \lambda \left[\frac{\beta^3}{\lambda} \frac{4\varepsilon}{(1+\varepsilon)^2} + \frac{\beta}{\lambda} \left(\frac{\beta^2}{1+\varepsilon} + \frac{\gamma^2}{1-\varepsilon} \right) \frac{1-5\varepsilon}{1+\varepsilon} + \beta\lambda(1-\varepsilon) \right] \\
&\quad - \frac{\beta}{1+\varepsilon} \left[2\beta^2 - \gamma^2 + \lambda^2(1-\varepsilon^2) - \left(\frac{\beta^2}{1+\varepsilon} + \frac{\gamma^2}{1-\varepsilon} \right) (1+3\varepsilon) \right] \\
&\quad - \frac{\gamma}{1-\varepsilon} (3\beta\gamma) \frac{1-\varepsilon}{1+\varepsilon} \\
&= 0.
\end{aligned}$$

To prove $JZ(\lambda) = 0$ we use (30), (31), and (38), and we find

$$\begin{aligned}
\lambda^{2/3} JZ(\lambda) &= \lambda JN(\lambda) - \frac{\beta}{1+\varepsilon} X(\lambda) - \frac{\gamma}{1-\varepsilon} JX(\lambda) \\
&= \lambda(-6\beta\gamma) \frac{\varepsilon}{1-\varepsilon^2} - \frac{\beta}{1+\varepsilon} (-3\gamma\lambda) \frac{1+\varepsilon}{1-\varepsilon} - \frac{\gamma}{1-\varepsilon} (3\beta\lambda) \frac{1-\varepsilon}{1+\varepsilon} \\
&= 0.
\end{aligned}$$

Since $JZ(\beta) = 0$, $JZ(\gamma) = 0$, and $JZ(\lambda) = 0$, it follows from Lemma 6 that $JZ(\alpha) = 0$ as well. □

We next define vectors,

$$T \stackrel{\text{def}}{=} +\frac{\beta}{1+\varepsilon} X + \frac{\gamma}{1-\varepsilon} JX + (\lambda^{1/3} - \lambda) JN = \frac{1}{\lambda^{1/3}} \left(\frac{\beta}{1+\varepsilon} Y + \frac{\gamma}{1-\varepsilon} JY \right),$$

$$S \stackrel{\text{def}}{=} -\frac{\gamma}{1-\varepsilon} X + \frac{\beta}{1+\varepsilon} JX,$$

so that $\{T, S, JZ\}$ is an orthogonal basis for the tangent space of M. (The second expression for T uses (49).) We take a cross-section $M' = M \cap \{(z_1, z_2) : y_1 = b\}$ for fixed $b \in \mathbb{R}$, and using a translation in the $\partial/\partial y_1$ direction we assume $b = 0$. Lemma 8 says that M is contained in the union of translates of M' provided the translates are taken in the $\partial/\partial y_1$ direction. We view M' as a surface in $\mathbb{R}^3$ where $\partial/\partial x_1$ is the vertical direction and $\partial/\partial x_2$ and $\partial/\partial y_2$ are the horizontal directions. Notice then that $\{T, S\}$ is an orthogonal basis for the tangent space of M' and T is horizontal.

The next lemma will permit us to see how M and M' are situated relative to the remaining coordinate directions.

Lemma 9. *Let $M^3 \subset \mathbb{C}^2$ be as described above with $\lambda \neq 0$. Consider the map $g : M \to \mathbb{C}^2$ defined according to*

$$g(p) = p - \frac{\gamma}{\lambda}\frac{1}{1-\varepsilon^2}Y + \frac{\beta}{\lambda}\frac{1}{1-\varepsilon^2}JY.$$

Then $T(g) = 0$ and $S(g)$ is parallel with Z.

Proof. We begin by giving simplified expressions for the partial derivatives of γ/λ and β/λ. From Lemmas 4, 5, and 6, and from (49), it follows that

$$\begin{aligned} X\left(\frac{\gamma}{\lambda}\right) &= +(1-\varepsilon^2)\lambda^{1/3}, \\ X\left(\frac{\beta}{\lambda}\right) &= 0, \\ JX\left(\frac{\gamma}{\lambda}\right) &= 0, \\ JX\left(\frac{\beta}{\lambda}\right) &= -(1-\varepsilon^2)\lambda^{1/3}, \\ JN\left(\frac{\gamma}{\lambda}\right) &= +(1-\varepsilon)\beta\lambda^{-2/3}, \\ JN\left(\frac{\beta}{\lambda}\right) &= -(1+\varepsilon)\gamma\lambda^{-2/3}. \end{aligned}$$

(The details are omitted.) Since Y and JY are constant, it then follows that

$$\begin{aligned} T(g) = &\frac{1}{\lambda^{1/3}}\left(\frac{\beta}{1+\varepsilon}Y + \frac{\gamma}{1-\varepsilon}JY\right) \\ &- \frac{1}{1-\varepsilon^2}\left[\frac{\beta}{1+\varepsilon}X\left(\frac{\gamma}{\lambda}\right) + \frac{\gamma}{1-\varepsilon}JX\left(\frac{\gamma}{\lambda}\right) + (\lambda^{1/3}-\lambda)JN\left(\frac{\gamma}{\lambda}\right)\right]Y \\ &+ \frac{1}{1-\varepsilon^2}\left[\frac{\beta}{1+\varepsilon}X\left(\frac{\beta}{\lambda}\right) + \frac{\gamma}{1-\varepsilon}JX\left(\frac{\beta}{\lambda}\right) + (\lambda^{1/3}-\lambda)JN\left(\frac{\beta}{\lambda}\right)\right]JY \\ = &\,0, \end{aligned}$$

as is easily checked. In addition,

$$\begin{aligned} S(g) = &-\frac{\gamma}{1-\varepsilon}X + \frac{\beta}{1+\varepsilon}JX - \frac{1}{1-\varepsilon^2}\left[-\frac{\gamma}{1-\varepsilon}X\left(\frac{\gamma}{\lambda}\right) + \frac{\beta}{1+\varepsilon}JX\left(\frac{\gamma}{\lambda}\right)\right]Y \\ &+ \frac{1}{1-\varepsilon^2}\left[-\frac{\gamma}{1-\varepsilon}X\left(\frac{\beta}{\lambda}\right) + \frac{\beta}{1+\varepsilon}JX\left(\frac{\beta}{\lambda}\right)\right]JY \\ = &-\frac{\gamma}{1-\varepsilon}X + \frac{\beta}{1+\varepsilon}JX + \frac{\gamma}{1-\varepsilon}\lambda^{-1/3}\left(\lambda X + \frac{\gamma}{1-\varepsilon}N + \frac{\beta}{1+\varepsilon}JN\right) \\ &- \frac{\beta}{1+\varepsilon}\lambda^{-1/3}\left(\lambda JX - \frac{\beta}{1+\varepsilon}N + \frac{\gamma}{1-\varepsilon}JN\right) \\ = &\,(1-\lambda^{2/3})\left(\lambda N - \frac{\gamma}{1-\varepsilon}X + \frac{\beta}{1+\varepsilon}JX\right) \\ = &\,(1-\lambda^{2/3})\,\lambda^{2/3}\,Z, \end{aligned}$$

where the next to last step also uses (49). □

Since T is horizontal, and since $T(g) = 0$ (by Lemma 9), it follows that each horizontal slice of $\mathbb{R}^3$ contains a unique point which is the image under g of the corresponding slice of M'. Ranging over all slices, the locus of such points is a curve that can be viewed as a graph over the x_1-axis. In fact, the curve is a straight line that is parallel with the x_1-axis. This uses the fact that $S(g)$ is parallel with Z (by Lemma 9) and the fact that S is independent of T. Following an additional translation in the horizontal directions, we may assume that $g(M')$ is contained in the x_1-axis.

We next determine the precise shape of these horizontal slices of M'. They are ellipses or hyperbolas, according to whether $\varepsilon < 1$ or $\varepsilon > 1$.

Lemma 10. *Defined on M, the point and line*

$$\begin{aligned} F_p &= g(p) + \sqrt{\frac{\alpha}{\lambda}}\frac{\sqrt{2\varepsilon}}{1-\varepsilon^2}JY \\ d_p &= \{g(p) + \sqrt{\frac{\alpha}{\lambda}}\frac{1}{(1-\varepsilon)\sqrt{2\varepsilon}}JY + sY : s \in \mathbb{R}\} \end{aligned}$$

are constant with respect to T. In addition, $\mathrm{dist}(p, F_p) = \sqrt{2\varepsilon/(1+\varepsilon)} \cdot \mathrm{dist}(p, d_p)$.

Proof. For the first claim, it is enough to verify that $T(\alpha/\lambda) = 0$, since Y and JY are constant, and since $T(g) = 0$ by Lemma 9. Using Lemma 6, together with the computations from the beginning of the proof of Lemma 9, we find

$$\begin{aligned}
T\left(\frac{\alpha}{\lambda}\right) &= T\left(\frac{1}{\lambda^2}\left(\frac{\beta^2}{1+\varepsilon}+\frac{\gamma^2}{1-\varepsilon}\right)\right)\\
&= \frac{2\beta}{\lambda}\frac{1}{1+\varepsilon}T\left(\frac{\beta}{\lambda}\right)+\frac{2\gamma}{\lambda}\frac{1}{1-\varepsilon}T\left(\frac{\gamma}{\lambda}\right)\\
&= \frac{2\beta}{\lambda}\frac{1}{1+\varepsilon}\left(\frac{\beta}{1+\varepsilon}\cdot 0-\frac{\gamma}{1-\varepsilon}(1-\varepsilon^2)\lambda^{1/3}-(\lambda^{1/3}-\lambda)(1+\varepsilon)\gamma\lambda^{-2/3}\right)\\
&\quad +\frac{2\gamma}{\lambda}\frac{1}{1-\varepsilon}\left(\frac{\beta}{1+\varepsilon}(1-\varepsilon^2)\lambda^{1/3}-\frac{\gamma}{1-\varepsilon}\cdot 0+(\lambda^{1/3}-\lambda)(1-\varepsilon)\beta\lambda^{-2/3}\right)\\
&= 0,
\end{aligned}$$

after an easy simplification. For the remaining claim, we find

$$\begin{aligned}
\operatorname{dist}(p,F_p)^2 &= \frac{1}{\lambda^2(1-\varepsilon^2)^2}\left(\gamma^2+\left(\beta+\sqrt{\alpha\lambda}\sqrt{2\varepsilon}\right)^2\right)\\
&= \frac{1}{\lambda^2(1-\varepsilon^2)^2}\left((1-\varepsilon)\left(\alpha\lambda-\frac{\beta^2}{1+\varepsilon}\right)+\left(\beta+\sqrt{\alpha\lambda}\sqrt{2\varepsilon}\right)^2\right)\\
&= \frac{1}{\lambda^2(1-\varepsilon^2)^2}\left(\beta\sqrt{2\varepsilon/(1+\varepsilon)}+\sqrt{\alpha\lambda}\sqrt{1+\varepsilon}\right)^2,
\end{aligned}$$

where we have again used Lemma 6, and

$$\operatorname{dist}(p,d_p)^2 = \frac{1}{\lambda^2(1-\varepsilon^2)^2}\left(\beta+\sqrt{\alpha\lambda}(1+\varepsilon)/\sqrt{2\varepsilon}\right)^2.$$

From these computations it follows that $\operatorname{dist}(p,F_p)=\sqrt{2\varepsilon/(1+\varepsilon)}\cdot\operatorname{dist}(p,d_p)$. □

In either case, relative to the horizontal coordinate $z_2 = x_2 + iy_2 = (x_2, y_2)$, the slice of M' has focus $F = (0, k\sqrt{2\varepsilon}/(1-\varepsilon^2))$ and directrix $d = \{y_2 = k/((1-\varepsilon)\sqrt{2\varepsilon})\}$, where $k = \sqrt{\alpha/\lambda}$ is constant in any slice. The ellipse or hyperbola has eccentricity $e = \sqrt{2\varepsilon/(1+\varepsilon)}$. (The cases $\varepsilon < 1$ and $\varepsilon > 1$ are illustrated in Figure

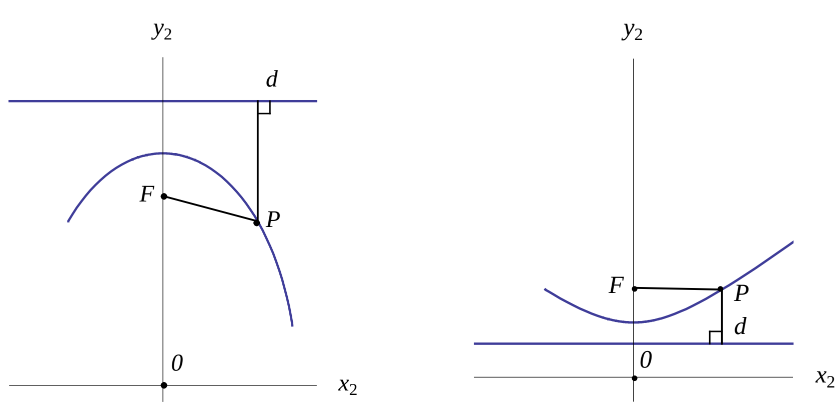

FIGURE 1. Horizontal cross-sections of M' for $\varepsilon = 1/2$ and $\varepsilon = 4$.

1.) Basic coordinate geometry can then be used to show that the slice of M' must satisfy

$$|z_2|^2 + \mathrm{Re}\,(\varepsilon z_2^2) = x_2^2(1+\varepsilon) + y_2^2(1-\varepsilon) = \frac{k^2}{1-\varepsilon^2}.$$

Putting everything together, after the uniform dilation, the special unitary transformation, and the translations, the original surface $M^3 \subset \mathbb{C}^2$ can be defined by

$$r(z_1, z_2) = \phi(z_1 + \overline{z}_1) + |z_2|^2 + \mathrm{Re}\,(\varepsilon z_2^2)$$

for some real function ϕ that can be assumed to be three-times differentiable. For this defining function we find that the condition $\mathcal{Q}_{r,p} = \varepsilon\,\mathcal{L}_{r,p}$ reduces to

$$\varepsilon(\phi')^2 + (\overline{z}_2 + \varepsilon z_2)^2\phi'' = \varepsilon\left((\phi')^2 + |z_2 + \varepsilon\overline{z}_2|^2\phi''\right),$$

and this ultimately requires that either $z_2 \equiv 0$ or $\phi'' \equiv 0$ on M. The case $z_2 \equiv 0$ is excluded, or else M is two-dimensional. So we conclude that $\phi'' \equiv 0$. Then, after a further translation in the x_1 variable, and a dilation restricted to the z_1 variable (which too is affine with real determinant), we conclude that M can be defined by $r(z_1, z_2) = (z_1 + \overline{z}_1) + |z_2|^2 + \mathrm{Re}\,(\varepsilon z_2^2)$. Theorem 1 is therefore proved.

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