# SPECTRUM OF THE KERZMAN-STEIN OPERATOR FOR THE ELLIPSE

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Abstract. The skew-hermitian part of the Cauchy operator, defined with respect to arclength measure on the boundary, is known as the Kerzman-Stein operator. For an ellipse, the eigenvalues of this operator are shown to have multiplicity two. For an ellipse with small eccentricity, we compute the leading coefficient in the asymptotic expansion of the eigenvalues.

## 1. Introduction

In their study of Cauchy-Fantappiè kernels and the Szegő kernel in higher dimensions, Kerzman and Stein discovered an elegant method for computing the Riemann map in one dimension. See [11, 12]. At the heart of their method is the fact that, for smooth, bounded domains, the Cauchy kernel and Szegő kernel have the same principal singularity at the diagonal. In particular, the skew-hermitian part of the Cauchy operator, called the Kerzman-Stein operator, is compact.

In a later article [10], Kerzman posed a number of problems concerning this operator, including the following.

**Problem** (Kerzman, 1979)**.** *Relate the spectrum of the Kerzman-Stein operator to the geometry of the domain.*

In a sense, the spectrum measures the error when the Cauchy kernel is used to approximate the Szegő kernel, and may be useful for estimating the rate of convergence of integral methods solutions to the Riemann map.

Here we provide an answer to the problem for the case of an ellipse with small eccentricity. In the following theorem, the ellipse has eccentricity $2\sqrt{\rho}/(1+\rho)$.

**Theorem.** *The Kerzman-Stein operator for an ellipse has eigenvalues $\pm i\lambda_l$ where each $\pm i\lambda_l$ has multiplicity 2. If the ellipse is parameterized by $t \to e^{it} + \rho e^{-it}$ with $0 < \rho < 1$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, then as $\rho \downarrow 0$,*

$$\lambda_l = \beta_l \cdot \rho^{2l-1} + o(\rho^{2l-1}) \ \text{ where } \ 0 < \beta_l \leq 1.$$

*The coefficients $\beta_l$ can be computed explicitly.*

Since the operator is compact and skew-hermitian, its spectrum will be discrete and imaginary, except for an accumulation point at zero. It is known already that

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the Kerzman-Stein operator vanishes identically for a circle. The ellipse, however, is the first simply-connected, smooth, bounded domain to be considered for this problem after the disc.

It is known as well that the Kerzman-Stein operator is invariant under Möbius transformations of the complex plane. So in a previous article [4] the author computed the spectrum for domains bounded by two circular arcs or two logarithmic spirals—logarithmic spirals are known to have constant inversive curvature. The ellipse, then, is the first example for which there is no apparent Möbius symmetry. See Wilker [19] or Coffman and Frantz [6] for more on this topic.

Formally, Kerzman's problem is similar to Schiffer's problem of computing the Fredholm eigenvalues of a plane domain [14, 15] which has been studied extensively. The similarity is explained by Burbea in [5], where he reduced Kerzman's eigenvalue problem to a problem previously studied by Singh [17]. The connection between Singh's problem and the Fredholm eigenvalue problem is apparent, for instance, in Bergman's book [3, p.71]. Loosely, Singh's problem can be interpreted as a boundary analogue of the Fredholm eigenvalue problem. For an ellipse, the Fredholm eigenvalues are known exactly—they are simply powers of the parameter $\rho$. For this fact, see [14, p.1195]. For more on the general problem, see also [2, 16].

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## 2. Preliminaries

We follow notation that Bell uses in his book [1]. Suppose $\Omega \subset \mathbb{C}$ is a bounded domain with twice differentiable boundary, and let $T = T(w)$ be the unit tangent vector at $w \in \partial\Omega$, oriented positively with respect to $\Omega$. If $ds$ is arclength measure on $\partial\Omega$, then the Kerzman-Stein operator is the operator defined by

$$Af(z) = \int_{\partial\Omega} A(z,w) f(w)\, ds_w = \frac{1}{2\pi i}\int_{\partial\Omega}\left(\frac{T(w)}{w-z} - \frac{\overline{T(z)}}{\overline{w}-\overline{z}}\right) f(w)\, ds_w,$$

valid for $f \in L^2(\partial\Omega)$ and $z \in \partial\Omega$. In fact, the kernel $A(z,w)$ is bounded at the diagonal—the apparent singularities cancel each other. The space $L^2(\partial\Omega)$ is defined using the hermitian inner product $(f,g)_{\partial\Omega} = \int_{\partial\Omega} f\overline{g}\, ds$.

Since $A(z,w)$ is bounded and $\partial\Omega$ has finite length, it follows that $A$ is Hilbert-Schmidt, and is therefore compact on $L^2(\partial\Omega)$. Since $A$ is also skew-hermitian, the spectral theorem says that its spectrum is discrete and bounded, and will consist of imaginary eigenvalues whose only accumulation point is zero. If $\Omega$ is unbounded or if $\partial\Omega$ has a corner, however, this is not necessarily true. See [4] for specific examples when $A$ is non-compact.

We point out two general facts about $A$ that will be helpful to our study of the ellipse. Let $\Delta$ be the unit disc, $\Delta = \{z : |z| < 1\}$.

**Lemma 1.** *Suppose $h : \partial\Delta \to \partial\Omega$ is biholomorphic in a neighborhood of $\partial\Delta$ and $\sqrt{h'}$ is single-valued near $\partial\Delta$. Then the Kerzman-Stein operator on $L^2(\partial\Omega)$ is* *unitary equivalent to the operator on* $L^2(\partial\Delta)$ *with kernel*

$$A(z,w) \;=\; \frac{1}{2\pi i}\left(\frac{iw\sqrt{h'(w)}\sqrt{h'(z)}}{h(w)-h(z)} - \frac{\overline{iz\sqrt{h'(w)}\sqrt{h'(z)}}}{\overline{h(w)}-\overline{h(z)}}\right) \text{ for } w,z\in\partial\Delta.$$

*Proof.* We first establish an isometry $L^2(\partial\Delta)\cong L^2(\partial\Omega)$ given by $(f\circ h)\sqrt{h'}\leftarrow f$. Taking $f,g\in L^2(\partial\Omega)$, this follows from

$$\int_{\partial\Omega} f\,\overline{g}\,ds = \int_{\partial\Delta}(f\circ h)\cdot\overline{(g\circ h)}\,|h'|\,ds = \int_{\partial\Delta}(f\circ h)\sqrt{h'}\cdot\overline{(g\circ h)\sqrt{h'}}\,ds.$$

Next, write for $f\in L^2(\partial\Omega)$,

$$A_{\partial\Omega}f(z') \;=\; \frac{1}{2\pi i}\int_{\partial\Omega}\left(\frac{T(w')}{w'-z'} - \frac{\overline{T(z')}}{\overline{w'}-\overline{z'}}\right)f(w')\,ds_{w'},$$

and replace $w'=h(w)$, $T(w')=\frac{h'(w)}{|h'(w)|}\cdot iw$, and $ds_{w'}=|h'(w)|\,ds_w$, and also $z'=h(z)$ and $T(z')=\frac{h'(z)}{|h'(z)|}\cdot iz$. Then,

$$\begin{aligned}&(A_{\partial\Omega}f)\circ h(z)\\ &=\frac{1}{2\pi i}\int_{\partial\Delta}\left(\frac{iw}{h(w)-h(z)}\frac{h'(w)}{|h'(w)|} - \frac{\overline{iz}}{\overline{h(w)}-\overline{h(z)}}\frac{\overline{h'(z)}}{|h'(z)|}\right)(f\circ h)(w)|h'(w)|\,ds_w.\end{aligned}$$

Multiplying both sides by $\sqrt{h'(z)}$ gives

$$\begin{aligned}&(A_{\partial\Omega}f)\circ h(z)\sqrt{h'(z)}\\ &=\frac{1}{2\pi i}\int_{\partial\Delta}\left(\frac{iw\sqrt{h'(w)}\sqrt{h'(z)}}{h(w)-h(z)} - \frac{\overline{iz\sqrt{h'(w)}\sqrt{h'(z)}}}{\overline{h(w)}-\overline{h(z)}}\right)(f\circ h)(w)\sqrt{h'(w)}\,ds_w,\end{aligned}$$

and the lemma is proved. □

**Lemma 2.** *The Kerzman-Stein operator commutes with the involution* $\Psi$ *on* $L^2(\partial\Omega)$ *given by* $f\to\overline{fT}$*; that is,* $A\Psi f=\Psi Af$*. So the (imaginary) spectrum of* $A$ *is symmetric with respect to 0.*

*Proof.* Notice that

$$\begin{aligned}A(z,w) \;&=\; \frac{1}{2\pi i}\left(\frac{T(w)}{w-z}-\frac{\overline{T(z)}}{\overline{w}-\overline{z}}\right) \;=\; \frac{1}{2\pi i}\left(\frac{\overline{T(z)}}{w-z}-\frac{\overline{T(w)}}{\overline{w}-\overline{z}}\right)T(w)\overline{T(z)}\\ &=\overline{\frac{1}{2\pi i}\left(-\frac{\overline{T(z)}}{\overline{w}-\overline{z}}+\frac{T(w)}{w-z}\right)}T(w)\overline{T(z)} \;=\; \overline{A(z,w)}\,T(w)\overline{T(z)},\end{aligned}$$

so

$$A\Psi f = \int_{\partial\Omega}A(z,w)\overline{f(w)T(w)}\,ds_w \;=\; \int_{\partial\Omega}\overline{A(z,w)}\,\overline{f(w)}\,ds_w\cdot\overline{T(z)} \;=\; \Psi Af.$$

So, if $Af=i\lambda f$, then also $A\Psi f=\Psi Af=\Psi(i\lambda f)=-i\lambda\Psi f$, and the lemma is proved. □

## 3. Approximation by Finite Rank Operators on the Unit Circle

The theorem is proved as follows. Using Lemma 1, we transform the problem from the ellipse to an equivalent problem on the disc. We express the new kernel using a double Fourier series, and approximate the operator using finite rank operators with kernels that have rapidly decaying coefficients. We then estimate the eigenvalues of the approximating operators. Finally, we use the Cauchy interlace theorem and a previous estimate of Feldman, Krupnik, and Spitkovsky in order to bound the leading coefficients.

3.1. **Pulling the Kerzman-Stein operator back to the unit disk.** For the ellipse, take $0 < \rho < 1$ and $h(z) = z + \rho/z$. Then, using the expansions

$$\sqrt{h'(z)} \;=\; \sqrt{1 - \frac{\rho}{z^2}} \;=\; \sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j} (-1)^j \frac{\rho^j}{z^{2j}}$$

and

$$h(w) - h(z) \;=\; (w + \frac{\rho}{w}) - (z + \frac{\rho}{z}) \;=\; (w - z)(1 - \frac{\rho}{wz}),$$

and also the identities $\overline{w} = 1/w$, $\overline{z} = 1/z$, and

$$\frac{\overline{iz}}{\overline{z} - \overline{w}} \;=\; \frac{-iw}{w - z}$$

for $w, z \in \partial\Delta$, we find by Lemma 1 that the Kerzman-Stein operator for the ellipse is unitary equivalent to the operator on $L^2(\partial\Delta)$ with kernel

$$A(z, w) \;=\; \frac{1}{2\pi} \frac{w}{w - z} \sum_{j,k,l \geq 0} (-1)^{j+k} \rho^{j+k+l} \binom{\frac{1}{2}}{j} \binom{\frac{1}{2}}{k} \big[ w^{-2j-l} z^{-2k-l} - w^{2j+l} z^{2k+l} \big].$$

The order of summation can be taken symmetrically with respect to $j$, $k$, $l$; that is, the partial sums are $s_n = \sum_{j+k+l \leq n}$. We will see that these partial sums are each divisible by $w - z$.

Next, identify $\partial\Delta$ with the interval $[0, 2\pi]$ and use the standard basis $\{\psi_j\}_{j \in \mathbb{Z}}$ for $L^2([0, 2\pi])$ given by $\psi_j(s) = e^{ijs}/\sqrt{2\pi}$. In terms of this basis, the Kerzman-Stein operator has kernel

$$\begin{aligned} A(t, s) \;=\; & \frac{1}{2\pi} \frac{e^{is}}{e^{is} - e^{it}} \cdot \\ & \sum_{j,k,l \geq 0} (-1)^{j+k} \rho^{j+k+l} \binom{\frac{1}{2}}{j} \binom{\frac{1}{2}}{k} \Big[ e^{-i(2j+l)s} e^{-i(2k+l)t} - e^{i(2j+l)s} e^{i(2k+l)t} \Big], \end{aligned}$$

or alternately,

$$A(t, s) \;=\; \sum_{j,k \in \mathbb{Z}} a_{j,k} \, \psi_j(t) \, \overline{\psi_k(s)}$$

for real coefficients $a_{j,k}$ that satisfy

(i) $a_{j,k} = 0$ if both $j, k < 0$, if both $j, k \geq 0$, or if $j+k$ is odd.

(ii) $a_{j,k} = \rho^{\frac{-j+k}{2}} b_{-j,k}$ if $j < 0 \le k$; the $b_{j,k}$ are determined by

$$\sum_{j>0,\, k\ge 0} b_{j,k}\, w^j z^k = \frac{w}{w-z} \sum_{\substack{j,k,l\ge 0\\ j+k+l>0}} (-1)^{j+k} \binom{\frac12}{j}\binom{\frac12}{k} w^{2j+l} z^{2k+l}.$$

(iii) $a_{j,k} = -a_{k,j}$ for all $j, k$.

(iv) $a_{j,k} = a_{-j-1,-k-1}$ for all $j, k$.

As an operator, $A$ then behaves as multiplication by the matrix $(a_{j,k})_{j,k\in\mathbb{Z}}$.

*Proof of (i)-(iv).* To prove (i), rewrite $A(z, w)$ using the expansion

$$\frac{1}{2\pi} \sum_{n>0} \rho^n \left(\frac{1}{w^{2n}}\frac{1}{z^{2n}} - 1\right) \frac{w}{w-z} \sum_{0\le l\le n} w^l z^l \sum_{\substack{j,k\ge 0\\ j+k=n-l}} (-1)^{j+k} \binom{\frac12}{j}\binom{\frac12}{k} w^{2j} z^{2k}$$

$$= \frac{1}{2\pi} \sum_{n>0} \rho^n \left(\frac{1}{w^{2n}}\frac{1}{z^{2n}} - 1\right) \frac{w}{w-z} \Bigg[\left(-\frac12 w^2 + wz - \frac12 z^2\right) w^{n-1} z^{n-1} + \sum_{0\le l\le n-2} w^l z^l \sum_{\substack{j,k\ge 0\\ j+k=n-l}} (-1)^{j+k} \binom{\frac12}{j}\binom{\frac12}{k} w^{2j} z^{2k}\Bigg].$$

We have assumed $n \overset{\text{def}}{=} j+k+l > 0$ since the terms for $n = 0$ cancel; we have also interchanged indices $j, k$ in the term $w^{-2j-l} z^{-2k-l}$ and factored out the quantity $(w^{-2n} z^{-2n} - 1)$. Since

$$\sum_{\substack{j,k\ge 0\\ j+k=m}} (-1)^{j+k} \binom{\frac12}{j}\binom{\frac12}{k} = 0$$

for $m \ge 2$, as shown in the next section, we may divide the quantity in brackets by $w - z$ (rather, we divide by $w^2 - z^2$, but $w - z$ is a factor of this) and obtain

$$\sum_{j,k\in\mathbb{Z}} a_{j,k}\, z^j w^{-k} = \sum_{n>0} \rho^n \left(\frac{1}{w^{2n}}\frac{1}{z^{2n}} - 1\right) \sum_{0\le l\le n-1} w^l z^l \sum_{1\le p\le 2(n-l)} c_{p,n-l}\, w^p\, z^{2(n-l)-p}$$

$$= \sum_{n>0} \sum_{\substack{0\le l\le n\\ 1\le p\le 2(n-l)}} \rho^n\, c_{p,n-l} \left[w^{-2n+l+p}\, z^{-l-p} - w^{l+p}\, z^{2n-l-p}\right],$$

for real coefficients $c_{p,m}$, $1 \le p \le 2m$, that can be given explicitly in terms of the binomial coefficients. The first terms on the right-hand side have the form $z^j w^{-k}$ for integers $j, k$ satisfying $j < 0$, $k \ge 0$, and $j + k$ even. The second terms have the form $z^j w^{-k}$ for integers $j, k$ satisfying $j \ge 0$, $k < 0$, and $j+k$ even. The coefficients $a_{j,k}$ for the remaining $j, k$ must be zero, so (i) is proved. To prove (ii), start with the $n$th degree terms in the definition of the $b_{j,k}$,

$$\sum_{\substack{j>0,\, k\ge 0\\ j+k=n}} b_{j,k}\, w^j z^k = \frac{w}{w-z} \sum_{\substack{j,k,l\ge 0\\ 2(j+k+l)=n}} (-1)^{j+k} \binom{\frac12}{j}\binom{\frac12}{k} w^{2j+l} z^{2k+l},$$

then multiply both sides by $\rho^{n/2}$ and sum on $n \geq 1$. After substituting $w = e^{is}$ and $z = e^{it}$, and replacing $j \to -j$ on the left-hand side we obtain

$$\sum_{\substack{j<0\\k\geq 0}} \rho^{\frac{-j+k}{2}} b_{-j,k}\, e^{-ijs} e^{ikt} = \frac{e^{is}}{e^{is}-e^{it}} \sum_{\substack{j,k,l\geq 0\\ j+k+l>0}} (-1)^{j+k} \rho^{j+k+l} \binom{\frac{1}{2}}{j}\binom{\frac{1}{2}}{k} e^{i(2j+l)s} e^{i(2k+l)t}$$

$$= -\sum_{k<0,\, j\geq 0} a_{j,k}\, e^{ijt} e^{-iks} = \sum_{k<0,\, j\geq 0} a_{k,j}\, e^{ijt} e^{-iks}.$$

The third equality uses (iii), which is proved next. Then, interchanging $j, k$ in the last expression gives $a_{j,k} = \rho^{\frac{-j+k}{2}} b_{-j,k}$ and (ii) is proved. To prove (iii), use the identity $e^{is}/(e^{is}-e^{it}) = e^{-it}/(e^{-it}-e^{-is})$ to verify that $\overline{A(s,t)} = -A(t,s)$. Then

$$\sum a_{j,k}\, \overline{\psi_j(s)}\, \psi_k(t) \;=\; -\sum a_{j,k}\, \psi_j(t)\, \overline{\psi_k(s)},$$

and after interchanging $j, k$, it follows that $a_{j,k} = -a_{k,j}$. To prove (iv), first verify that $\overline{A(t,s)} = A(t,s)e^{-is}e^{it}$. Then

$$\sum a_{j,k}\, \overline{\psi_j(t)}\, \psi_k(s) \;=\; \sum a_{j,k}\, \psi_{j+1}(t)\, \overline{\psi_{k+1}(s)} \;=\; \sum a_{j-1,k-1}\, \psi_j(t)\, \overline{\psi_k(s)}$$

$$= \sum a_{-j-1,-k-1}\, \overline{\psi_j(t)}\, \psi_k(s).$$

The second equality uses the replacements $j \to j-1$, $k \to k-1$, and the third equality uses the replacements $j \to -j$, $k \to -k$, and the fact that $\psi_{-j} = \overline{\psi_j}$. □

These properties enable us to show the eigenspaces of $A$ are even-dimensional.

**Lemma 3.** *If the Kerzman-Stein operator has an eigenvalue $i\lambda$ that corresponds with eigenvector $v = v^{1,+} = \sum v_j \psi_j$, then it has a second eigenvector $v^{2,+} = \sum v_{-j-1}\psi_j$ that corresponds with $i\lambda$. There are also eigenvectors $v^{1,-} = \sum \overline{v}_j \psi_j$ and $v^{2,-} = \sum \overline{v}_{-j-1}\psi_j$ that correspond with eigenvalue $-i\lambda$.*

*Proof.* Suppose that $v^{1,+} = \sum_j v_j\psi_j$ so $Av^{1,+} = A(\sum_j v_j\psi_j) = \sum_{j,k} a_{j,k}v_k\,\psi_j$ and $\sum_k a_{j,k}v_k = i\lambda v_j$ for each $j$. Then, using (iv),

$$Av^{2,+} = \sum_{j,k} a_{j,k} v_{-k-1}\psi_j = \sum_{j,k} a_{-j-1,-k-1} v_{-k-1}\psi_j = \sum_j i\lambda v_{-j-1}\psi_j = i\lambda v^{2,+}.$$

A parity argument shows that $v^{1,+}$ and $v^{2,+}$ can be made independent of one another. To see this, decompose

$$v \;=\; v_{\text{odd}} + v_{\text{even}} \stackrel{\text{def}}{=} \sum_{j \text{ odd}} v_j\psi_j + \sum_{j \text{ even}} v_j\psi_j.$$

If $v_{\text{odd}} \neq 0$, then replace $v = v^{1,+} = v_{\text{odd}}$; else replace $v^{1,+} = v_{\text{even}}$. Then, $Av^{1,+} = i\lambda v^{1,+}$ still holds, since $a_{j,k} = 0$ for $j+k$ odd. Moreover, $v^{1,+} = \sum_j v_j\psi_j$ and $v^{2,+} = \sum_j v_{-j-1}\psi_j$ will have opposite odd/even parity, and will therefore be orthogonal. Finally, we describe the opposite eigenspace. If $v^{1,-} = \sum_j \overline{v}_j\psi_j$, we find

$$Av^{1,-} \;=\; \sum_{j,k} a_{j,k}\overline{v}_k\psi_j \;=\; \sum_{j,k} \overline{a}_{j,k}\overline{v}_k\psi_j \;=\; \sum_j -i\lambda\overline{v}_j\psi_j \;=\; -i\lambda v^{1,-}.$$

Then also $v^{2,-} = \sum \overline{v}_{-j-1}\psi_j$ is an eigenvector corresponding with $-i\lambda$, since it is gotten from $v^{1,-}$ in the same way that $v^{2,+}$ is gotten from $v^{1,+}$. □

From now on, we associate the spectrum of $A$ with a list of values $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, so the eigenvalues of $A$ are $\pm i\lambda_j$, where each $\pm i\lambda_j$ has multiplicity 2.

3.2. **Approximation by finite rank operators.** We now pass to a set of finite rank operators $A_n$ with degenerate kernel

$$A_n(t,s) \;=\; \sum_{j,k=-2n}^{2n-1} a_{j,k}\, \psi_j(t)\, \overline{\psi_k(s)}.$$

The rank of $A_n$ is at most $4n$. Furthermore, Lemma 3 also holds for the $A_n$ since the span of $\psi_{-2n}, \ldots, \psi_{2n-1}$ is unchanged under the transformation $\psi_j \to \psi_{-j-1}$. We then associate the spectrum of $A_n$ with a list of values $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, so that the eigenvalues of $A_n$ are $\pm i\lambda_j$ and each $\pm i\lambda_j$ has multiplicity 2.

As an operator, $A_n$ acts as multiplication by the matrix $(a_{j,k})_{j,k=-2n,\ldots,2n-1}$. To illustrate this, we show the matrix for $A_3$ in Figure 1. The columns and rows are indexed over $-6, -5, \ldots, +5$. Notice that the matrix is skew-hermitian (in

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{-\rho^3}{16} & 0 & \frac{-25\rho^4}{128} & 0 & \frac{-81\rho^5}{128} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-3\rho^3}{16} & 0 & \frac{-81\rho^4}{128} & 0 & \frac{81\rho^5}{128} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-\rho^2}{8} & 0 & \frac{-5\rho^3}{8} & 0 & \frac{81\rho^4}{128} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-5\rho^2}{8} & 0 & \frac{5\rho^3}{8} & 0 & \frac{25\rho^4}{128} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-\rho}{2} & 0 & \frac{5\rho^2}{8} & 0 & \frac{3\rho^3}{16} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho}{2} & 0 & \frac{\rho^2}{8} & 0 & \frac{\rho^3}{16} \\
\frac{\rho^3}{16} & 0 & \frac{\rho^2}{8} & 0 & \frac{\rho}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3\rho^3}{16} & 0 & \frac{5\rho^2}{8} & 0 & \frac{-\rho}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{25\rho^4}{128} & 0 & \frac{5\rho^3}{8} & 0 & \frac{-5\rho^2}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{81\rho^4}{128} & 0 & \frac{-5\rho^3}{8} & 0 & \frac{-\rho^2}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{81\rho^5}{128} & 0 & \frac{-81\rho^4}{128} & 0 & \frac{-3\rho^3}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-81\rho^5}{128} & 0 & \frac{-25\rho^4}{128} & 0 & \frac{-\rho^3}{16} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

FIGURE 1. The matrix associated to $A_3$

fact, real skew-symmetric) and the powers of $\rho$ increase away from the diagonal. This illustrates properties (ii) and (iii) of the $a_{j,k}$. Properties (i) and (iv) are also apparent. The innermost $4 \times 4$ and $8 \times 8$ submatrices correspond with $A_1$ and $A_2$, respectively.

The next result assumes that $\rho \downarrow 0$, so we are considering only ellipses with small eccentricity.

**Lemma 4.** *As described above, associate the spectrum of $A_n$ with the list of values $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Then, for $1 \leq l \leq n$,*

$$\lambda_l = \beta_l \cdot \rho^{2l-1} + o(\rho^{2l-1}) \textit{ for constants } \beta_l > 0.$$

*The coefficients $\beta_l$ can be computed explicitly.*

*Proof.* Consider first the characteristic polynomial of the matrix associated to $A_n$,

$$p(\lambda) \;=\; \det(A_n - \lambda I) \;=\; \sum_{\sigma} \prod_{j=-2n}^{2n-1} (-1)^{\sigma}\,(a_{j,\sigma_j} - \lambda \delta_{j,\sigma_j}) \;=\; \sum_{j=0}^{4n} (-1)^j \alpha_j\, \lambda^{4n-j},$$

where $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$. The sum is taken over permutations of $\{-2n, \ldots, 2n-1\}$ and $(-1)^{\sigma}$ indicates the signature of a permutation $\sigma$.

The zeros of $p(\lambda)$ are the eigenvalues of $A_n$, and the coefficients $\alpha_j$ are the *sums of products of eigenvalues taken* $j$ *at a time.* Consider specifically the coefficient $\alpha_{4l}$ in the expression. A permutation contributes a term to this coefficient only if it acts identically on $4n - 4l$ entries. (There are zeros along the diagonal of matrix $(a_{j,k})$.) Of these permutations, only those that act identically away from $\{-2l, \ldots, 2l-1\}$ contribute a term with factor $\rho^{4l^2}$. The others contribute higher powers of $\rho$; this uses (ii). In fact, the permutations that act identically away from $\{-2l, \ldots, 2l-1\}$ contribute $\det A_l$ to the coefficient $\alpha_{4l}$, and $\alpha_{4l} = \det A_l + o(\rho^{4l^2})$.

Next, since $A_l$ is skew-symmetric and zero in its upper-left and lower-right quadrants, its determinant is the square of the determinant of its upper-right quadrant. The entries in the upper-right quadrant are $\rho^{\frac{-j+k}{2}}\, b_{-j,k}$, so $\det A_l = \rho^{4l^2} (\det B_l)^2$ where $B_l$ is the matrix

$$B_l \;=\; (b_{j,k})_{\substack{j=1,\ldots,2l \\ k=0,\ldots,2l-1}} \,.$$

(This also involves interchanging an even number of rows, but this doesn't affect the determinant.) It follows that $\alpha_{4l} = \rho^{4l^2} (\det B_l)^2 + o(\rho^{4l^2})$.

Proposition 1, stated and proved in the next section, says that $\det B_l \neq 0$ for each $l$. So then $\alpha_{4l}$ is always comparable to $\rho^{4l^2}$. In particular, when $l = 1$, the product of the four largest eigenvalues is comparable to $\rho^4$. That is,

$$\lambda_1^4 \;=\; (+i\lambda_1)(+i\lambda_1)(-i\lambda_1)(-i\lambda_1) \;=\; \beta_1' \cdot \rho^4 + o(\rho^4), \tag{1}$$

for $\beta_1' \neq 0$. In general, the product of the $4l$ largest eigenvalues is comparable to $\rho^{4l^2}$. That is,

$$\lambda_1^4\, \lambda_2^4 \cdots \lambda_l^4 \;=\; \beta_l' \cdot \rho^{4l^2} + o(\rho^{4l^2}), \tag{2}$$

for $\beta_l' \neq 0$. From (1) and (2) it follows inductively that $\lambda_l = \beta_l\, \rho^{2l-1} + o(\rho^{2l-1})$ for $\beta_l \neq 0$. Next, since $\lambda_l = O(\rho^{2l-1})$ for each $l$, we conclude that $\alpha_{4l}$ and $\lambda_1^4\, \lambda_2^4 \cdots \lambda_l^4$ agree to order $\rho^{4l^2}$, and

$$\lambda_1^4\, \lambda_2^4 \cdots \lambda_l^4 \;=\; \rho^{4l^2} (\det B_l)^2 + o(\rho^{4l^2}). \tag{3}$$

It now follows inductively from (3) that $\beta_1 = (\det B_1)^{1/2}$, and for $l > 1$, $\beta_l = (\det B_l)^{1/2} \cdot (\det B_{l-1})^{-1/2}$. So the lemma is proved. $\square$

Using a result of Hermann Weyl [18], we recover the same information for $A$ that we now have for $A_n$. Weyl proved the result for the case of symmetric kernels; the proof is the same for the case of hermitian or skew-hermitian kernels. See also Porter and Stirling [13, p.146-147].

**Lemma 5.** *Suppose $K'$ and $K''$ are symmetric kernels in $L^2[(a,b)\times(a,b)]$ and denote by $\chi_1, \chi_2, \ldots$, the eigenvalues for any such kernel, repeated according to multiplicity, and arranged so that $|\chi_1| \geq |\chi_2| \geq \cdots$. Then,*

$$|\chi_{j+k+1}(K'+K'')| \;\leq\; |\chi_{j+1}(K')| + |\chi_{k+1}(K'')|.$$

Using the lemma, set $K' = A_n$, $K'' = A - A_n$, $j = l-1$, and $k = 0$. Then

$$|\chi_l(A)| \leq |\chi_l(A_n)| + |\chi_1(A - A_n)|.$$

Also, if $K' = A$ and $K'' = A_n - A$ then

$$|\chi_l(A_n)| \leq |\chi_l(A)| + |\chi_1(A_n - A)|.$$

This means that the eigenvalues of $A$ agree with the eigenvalues of $A_n$ to within $|\chi_1(A - A_n)|$. So by (ii), they agree to within $O(\rho^{n+1})$, since this is the Hilbert-Schmidt norm of $A - A_n$.

Returning to the notation with eigenvalues $\pm i\lambda_l$ arranged with $\lambda_1 \geq \lambda_2 \geq \cdots$, where each $\pm i\lambda_l$ has multiplicity 2, we now choose $n \geq 2l-1$ and find

$$\lambda_l(A) = \lambda_l(A_n) + O(\rho^{2l}) = \beta_l \cdot \rho^{2l-1} + o(\rho^{2l-1}),$$

where the $\beta_l \neq 0$ are specified in the proof of Lemma 4. Apart from Proposition 1, then, we have left to show only that the $\beta_l$ are bounded by 1.

3.3. **Bounding the coefficients.** We start with a variant of the Cauchy interlace theorem. Following our earlier convention, associate the spectra of $A_l$ and $A_{l-1}$ with values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l-1} \geq 0$, respectively. Then, the relevant minimax formula for $\lambda_{j+1}$ is

$$\lambda_{j+1} \;=\; \min_{u_1,\ldots,u_j} \left[ \max_v \frac{|(v, A_l v)|}{(v,v)} \right],$$

where the maximum is taken over vectors $v$ orthogonal to vectors $u_1, \ldots, u_j$ *and their images under the transformations*

$$\sum v_k \psi_k \;\to\; \sum v_{-k-1}\psi_k,\; \sum \overline{v}_k \psi_k,\; \sum \overline{v}_{-k-1}\psi_k$$

that occur in Lemma 3. Notice that $A_l$ is gotten from $A_{l-1}$ by adding 4 rows and columns, but these rows and columns correspond with a single vector (for instance, $v = \psi_{2l-1} + i\psi_{2l-2}$) and its transformations. Then, following the usual proof of the interlace theorem as given by Franklin [8], for instance, it follows that

$$\lambda_1 \;\geq\; \mu_1 \;\geq\; \lambda_2 \;\geq\; \mu_2 \;\geq\; \cdots \;\geq\; \mu_{l-1} \;\geq\; \lambda_l \;>\; 0.$$

Next, we use Feldman, Krupnik, and Spitkovsky's estimate [7] that says $\|A\| < 1$ for any ellipse, and in particular, $\lambda_1 < 1$. Then

$$\det A_l \;=\; (\lambda_1\lambda_2\cdots\lambda_l)^4 \;<\; (1\cdot\lambda_2\cdots\lambda_l)^4 \;\leq\; (1\cdot\mu_1\cdots\mu_{l-1})^4 \;=\; \det A_{l-1}.$$

This is true for any $\rho$, so letting $\rho \uparrow 1$ gives

$$(\det B_l)^2 \;=\; \big(\det(b_{j,k})_{\substack{j=1,\ldots,2l\\ k=0,\ldots,2l-1}}\big)^2 \;\leq\; \big(\det(b_{j,k})_{\substack{j=1,\ldots,2l-2\\ k=0,\ldots,2l-3}}\big)^2 \;=\; (\det B_{l-1})^2,$$

and therefore, $\beta_l \leq 1$.

Numerically, it seems that the coefficients $\beta_l$ increase with an upper limit of 1. Figure 2 illustrates this behavior for $1 \le l \le 40$. The first few leading coefficients

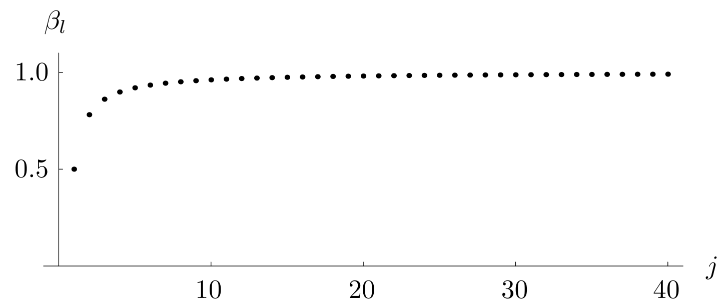


FIGURE 2. Plot of the leading coefficients, $\beta_l$, for $1 \le l \le 40$.

are $\beta_1 = 1/2$, $\beta_2 = 25/32$, $\beta_3 = 441/512$, and $\beta_4 = 184041/204800$. The author determined these coefficients using Mathematica.

## 4. Proof of Proposition 1

The definition of the $b_{j,k}$ in the proposition below is slightly different from the previous section—we have multiplied the right-hand side by $z$. So the coefficients $b_{j,k}$ for $1 \le j, k \le 2m$ in this section correspond with the coefficients $b_{j,k}$ for $1 \le j \le 2m$, $0 \le k \le 2m-1$ in the previous section. The order of summation can again be taken symmetrically with respect to $w$ and $z$; for instance, use partial sums $\sum_{j+k\le n}$ for the left-hand side, and partial sums $\sum_{j+k+l\le n}$ for the right-hand side.

**Proposition 1.** *Let coefficients $b_{j,k}$ be determined by the equation*

$$\sum_{j,k=1\ldots\infty} b_{j,k}\, w^j z^k = \frac{wz}{w-z} \sum_{\substack{j,k,l=0\ldots\infty \\ j+k+l>0}} (-1)^{j+k} \binom{\frac{1}{2}}{j} \binom{\frac{1}{2}}{k} w^{2j+l} z^{2k+l}.$$

*If $B_m$ is the matrix $(b_{j,k})_{j,k=1\ldots 2m}$, then $\det B_m \neq 0$ for $m > 0$.*

The proof is combinatorial in nature, and depends on the fact that the $b_{j,k}$ are dyadic rational; that is, they are either zero or they have the form $r = 2^n \cdot u$ for $u$ odd and $n \in \mathbb{Z}$. We use the valuation $|\cdot|_2$ defined on all rationals according to

$$|r|_2 = \begin{cases} 2^{-n} & \text{if } r \neq 0 \\ 0 & \text{if } r = 0 \end{cases}$$

where in the first case $r = 2^n \cdot u/v$ for odd $u, v$. For example, $|5/8|_2 = 8$. The valuation has the following properties (see Jacobson [9, p.211]):

(a) $|r_1 r_2|_2 = |r_1|_2 \cdot |r_2|_2$
(b) $|r_1 + r_2|_2 \le \max(|r_1|_2, |r_2|_2)$, with equality only if $|r_1|_2 \neq |r_2|_2$ or $r_1 = r_2 = 0$

The proof of Proposition 1 is structured as follows. Let $\sigma$ range over all permutations of $\{1,\dots,2m\}$. Then, since $\det B_m = \sum_\sigma \prod_{j=1\dots 2m} (-1)^\sigma b_{j,\sigma_j}$, we have

$$|\det B_m|_2 \le \max_\sigma |\prod_{j=1\dots 2m} b_{j,\sigma_j}|_2.$$

This will be an equality provided there is a unique permutation $\sigma$ for which the maximum is attained. We will find such a maximizing permutation, then $|\det B_m|_2 \neq 0$ and $\det B_m \neq 0$. That is, we will show $\det B_m$ is nonzero dyadic rational.

We begin by introducing intermediate coefficients $d_{j,k}$ defined by

$$\sum_{j,k=0\dots\infty} d_{j,k}\, w^j z^k = wz + \sum_{\substack{j,k=0\dots\infty \\ j+k>0}} (-1)^{j+k} \binom{\frac{1}{2}}{j}\binom{\frac{1}{2}}{k} w^{2j} z^{2k},$$

and we claim that if $n$ is fixed, then $\sum_{j+k=n} d_{j,k} = 0$. For $n=2$, it is easy to check that $d_{0,2}+d_{1,1}+d_{2,0} = (-1/2)+1+(-1/2) = 0$. For larger $n$, notice that $d_{j,k}=0$ if either $j$ or $k$ is odd. In particular, if $n$ is odd then $\sum_{j+k=n} d_{j,k} = 0$. There remains the case when $n=2p$ for $p \ge 2$. We find that

$$\sum_{j+k=n} d_{j,k} = \sum_{j+k=p} (-1)^{j+k} \binom{\frac{1}{2}}{j}\binom{\frac{1}{2}}{k},$$

and this is the coefficient of $x^p$ in the expansion

$$1-x = \sqrt{1-x}\cdot\sqrt{1-x} = \sum_{j,k\ge 0} (-1)^{j+k} \binom{\frac{1}{2}}{j}\binom{\frac{1}{2}}{k} x^{j+k},$$

which is evidently zero. So the claim is proved.

We may then divide by $w-z$ (rather, we divide by $w^2-z^2$, but $w-z$ is a factor of this) and obtain coefficients $c_{j,k}$ that satisfy

$$\sum_{j,k=1\dots\infty} c_{j,k}\, w^j z^k = \frac{w\,z}{w-z} \sum_{j,k=0\dots\infty} d_{j,k}\, w^j z^k.$$

The $c_{j,k}$ and $d_{j,k}$ are related by

$$c_{n,1} = d_{n,0} \text{ and } c_{n-j,j+1} = d_{n-j,j} + c_{n-j+1,j} \text{ for } 1 \le j < n. \tag{4}$$

Notice in particular that $c_{2,1} = -1/2$ and $c_{1,2} = 1/2$.

Following Jacobson [9, p.211], define $\nu(r) = -\log_2 |r|_2$ and $\nu(0) = \infty$. For example, $\nu(5/8) = -\log_2 8 = -3$. Then the following properties of $\nu$ are equivalent to the corresponding properties of $|\cdot|_2$; we will use both repeatedly:

(a) $\nu(r_1 r_2) = \nu(r_1) + \nu(r_2)$
(b) $\nu(r_1+r_2) \ge \min(\nu(r_1),\nu(r_2))$, with equality only if $\nu(r_1) \neq \nu(r_2)$ or $r_1 = r_2 = 0$

Observe that if $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$, then $\nu(\binom{1/2}{j}) = -\sum_{p\ge 0} \lfloor j/2^p \rfloor$ for $j \ge 0$, since

$$\binom{\frac{1}{2}}{j} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{1}{2}-j+1)}{j!}.$$

On the right-hand side of this identity, the $j$ multiplicative factors each contribute a factor of $1/2$, and the $j!$ contributes $\lfloor j/2\rfloor + \lfloor j/4\rfloor + \lfloor j/8\rfloor + \cdots$ factors of $1/2$ since $\lfloor j/2^k\rfloor$ is the number of multiples of $2^k$ in $1, 2, \ldots, j$.

Using this, we can compute the values $\nu(d_{j,k})$ and obtain a lower estimate for the values $\nu(c_{j,k})$.

**Lemma 6.** *If $1 \le j \le n$, then $\nu(c_{j,n+1-j}) \ge -\sum_{p>0}\lfloor n/2^p\rfloor$. There is equality when $n$ is a positive power of 2.*

*Proof.* If $n$ is odd then $d_{j,n-j} \equiv 0$, so $c_{j,n+1-j} \equiv 0$ and $\nu(c_{j,n+1-j}) \equiv \infty$, and if $n = 2$ we see directly that $\nu(c_{2,1}) = \nu(c_{1,2}) = -1$. So suppose $n \ge 4$ is even, then $d_{j,n-j} = 0$ if $j$ is odd, and $d_{j,n-j} = (-1)^{n/2}\binom{1/2}{j/2}\binom{1/2}{(n-j)/2}$ if $j$ is even. So if $j$ is even,

$$\nu(d_{j,n-j}) = -\sum_{p\ge 0}\Big(\Big\lfloor \frac{\frac{j}{2}}{2^p}\Big\rfloor + \Big\lfloor \frac{\frac{n-j}{2}}{2^p}\Big\rfloor\Big) \ge -\sum_{p>0}\Big\lfloor \frac{n}{2^p}\Big\rfloor,$$

and since the $c_{j,n+1-j}$ are combinations of the $d_{j,n-j}$ by (4) we have proved the first part of the lemma.

If $n = 2^q$ for $q \ge 1$, we show there is equality. First, notice that $\nu(d_{j,2^q-j}) = \nu(0) = \infty$ if $j$ is odd. Next,

$$\nu(d_{2^q,0}) = -\sum_{p\ge 0}\Big(\Big\lfloor \frac{\frac{2^q}{2}}{2^p}\Big\rfloor + \Big\lfloor \frac{0}{2^p}\Big\rfloor\Big) = -\sum_{p>0}\Big\lfloor \frac{2^q}{2^p}\Big\rfloor,$$

and for $0 < j < 2^q$, $j$ even,

$$\nu(d_{2^q-j,j}) = -\sum_{p\ge 0}\Big(\Big\lfloor \frac{\frac{2^q-j}{2}}{2^p}\Big\rfloor + \Big\lfloor \frac{\frac{j}{2}}{2^p}\Big\rfloor\Big) > -\sum_{p\ge 0}\Big\lfloor \frac{\frac{2^q}{2}}{2^p}\Big\rfloor = -\sum_{p>0}\Big\lfloor \frac{2^q}{2^p}\Big\rfloor.$$

Here the inequality is strict because when $p = q - 1$, then

$$\Big\lfloor \frac{\frac{2^q-j}{2}}{2^p}\Big\rfloor + \Big\lfloor \frac{\frac{j}{2}}{2^p}\Big\rfloor \;=\; 0 + 0 \;<\; 1 = \Big\lfloor \frac{\frac{2^q}{2}}{2^p}\Big\rfloor.$$

Using (4), it then follows for $1 \le j \le 2^q$ that $|c_{j,2^q+1-j}|_2 = -\sum_{p>0}\lfloor 2^q/2^p\rfloor$, and the lemma is proved. $\square$

Next we show that $b_{j,k} = \sum_{l=0\dots\min(j-1,k-1)} c_{j-l,k-l}$. This follows from

$$\sum_{j,k=1\dots\infty} \sum_{l=0}^{\min(j-1,k-1)} c_{j-l,k-l}\, w^j z^k = \sum_{l=0}^{\infty} \sum_{j,k=l+1\dots\infty} c_{j-l,k-l}\, w^{j-l} z^{k-l} \cdot w^l z^l$$
$$= \sum_{l=0}^{\infty} \sum_{j,k=1\dots\infty} c_{j,k}\, w^j z^k \cdot w^l z^l$$
$$= \sum_{l=0}^{\infty} \frac{w\,z}{w-z} \sum_{j,k=0\dots\infty} d_{j,k}\, w^j z^k\, w^l z^l$$
$$= \sum_{l=0}^{\infty} \frac{w\,z}{w-z} \left[ w\,z + \sum_{\substack{j,k=0\dots\infty \\ j+k>0}} (-1)^{j+k} \binom{\frac{1}{2}}{j} \binom{\frac{1}{2}}{k} w^{2j} z^{2k} \right] w^l z^l$$
$$= \frac{w\,z}{w-z} \sum_{\substack{j,k,l=0\dots\infty \\ j+k+l>0}} (-1)^{j+k} \binom{\frac{1}{2}}{j} \binom{\frac{1}{2}}{k} w^{2j+l} z^{2k+l}.$$

We now estimate the value of $\nu$ on the $b_{j,k}$.

**Lemma 7.** *If $1 \le j \le n$, then $\nu(b_{j,n+1-j}) \ge -\sum_{p>0} \lfloor n/2^p \rfloor$. There is equality when $n$ is a positive power of 2.*

*Proof.* Using Lemma 6 we compute

$$\nu(b_{j,n+1-j}) = \nu\Big( \sum_{l=0}^{\min(j-1,n-j)} c_{j-l,n+1-j-l} \Big) \ge \min_{l=0\dots\min(j-1,n-j)} \nu(c_{j-l,n+1-j-l})$$
$$\ge \min_{l=0\dots\min(j-1,n-j))} -\sum_{p>0} \lfloor \frac{n-2l}{2^p} \rfloor = -\sum_{p>0} \lfloor \frac{n}{2^p} \rfloor.$$

Furthermore, if $n = 2^q$ for $q \ge 1$, then for $l \ne 0$,

$$\nu(c_{j-l,2^q+1-j-l}) \ge -\sum_{p>0} \lfloor \frac{2^q-2l}{2^p} \rfloor > -\sum_{p>0} \lfloor \frac{2^q}{2^p} \rfloor = \nu(c_{j,2^q+1-j}).$$

So we find that

$$\nu(b_{j,2^q+1-j}) = \nu\Big( \sum_{l=0}^{\min(j-1,2^q-j)} c_{j-l,2^q+1-j-l} \Big)$$
$$= \min_{l=0\dots\min(j-1,2^q-j)} \nu(c_{j-l,2^q+1-j-l}) = \nu(c_{j,2^q+1-j}) = -\sum_{p>0} \lfloor \frac{2^q}{2^p} \rfloor,$$

and the lemma is proved. □

We come now to the proof of the proposition, and as a matter of notation set $\nu^*_n = \sum_{p>0} \lfloor n/(2^p) \rfloor$. Then the previous lemma says $\nu(b_{j,k}) \ge -\nu^*_{j+k-1}$ for any $j,k$ with equality provided $j+k-1$ is a positive power of 2. We claim *there is a unique permutation $\sigma$ of $\{1,\dots,2m\}$ that maximizes $\sum_{j=1}^{2m} \nu^*_{j+\sigma_j-1}$, and for this* *permutation, $j+\sigma_j-1$ is a positive power of 2 for all $j$.* The proposition then follows since

$$\sum_{j=1\ldots 2m}\nu(b_{j,\tau_j}) \geq -\sum_{j=1\ldots 2m}\nu^*_{j+\tau_j-1} > -\sum_{j=1\ldots 2m}\nu^*_{j+\sigma_j-1} = \sum_{j=1\ldots 2m}\nu(b_{j,\sigma_j})$$

for any other permutation $\tau$. In particular, $\nu(\det B_m) = \sum_{j=1\ldots 2m}\nu(b_{j,\sigma_j}) \neq \infty$, and $\det B_m \neq 0$.

We describe how $\sigma$ is chosen. As $j+k-1$ increases through the even numbers, there is a significant drop in the value of $\nu(b_{j,k})$ when $j+k-1$ reaches a power of 2; in fact, the size of the drop increases with successive powers of 2. So working from the lower-right of the matrix, choose the permutation $\sigma$ so that $j+\sigma_j-1$ is the largest possible power of 2. This determines $\sigma_j$ for the range $2^q-2m < j \leq 2m$, where $2^{q-1} < 2m \leq 2^q$. In particular, $\sigma$ restricts to a permutation of $\{2^q-2m+1,\ldots,2m\}$. Now repeat the procedure for the remaining $(2^q-2m)\times(2^q-2m)$ upper-left submatrix and continue. This procedure uniquely determines a permutation $\sigma$ and the claim says that $\sigma$ uniquely minimizes $\sum_{j=1\ldots 2m}\nu(b_{j,\sigma_j}) = -\sum_{j=1\ldots 2m}\nu^*_{j+\sigma_j-1}$.

To be precise, if $\sigma^{2m}$ is the optimal permutation for $\{1,\ldots,2m\}$ and if $q$ is chosen so that $2^{q-1} < 2m \leq 2^q$, we will find that

$$\sigma_j^{2m} = \begin{cases} 2^q - j + 1 & \text{if } 2^q-2m < j \leq 2m \\ \sigma_j^{2^q-2m} & \text{otherwise.} \end{cases}$$

As an example, in Figure 3 we show the matrices with entries $\nu(b_{j,k})$ and $\nu^*_{j+k-1}$ for $1 \leq j,k \leq 2m$, where $m=3$. (See Figure 1 for the related matrix $A_3$.) The

$$\begin{pmatrix} \infty & \mathbf{-1} & \infty & -3 & \infty & -4 \\ \mathbf{-1} & \infty & -3 & \infty & -4 & \infty \\ \infty & -3 & \infty & -3 & \infty & \mathbf{-7} \\ -3 & \infty & -3 & \infty & \mathbf{-7} & \infty \\ \infty & -4 & \infty & \mathbf{-7} & \infty & -7 \\ -4 & \infty & \mathbf{-7} & \infty & -7 & \infty \end{pmatrix} \qquad \begin{pmatrix} 0 & \mathbf{1} & 1 & 3 & 3 & 4 \\ \mathbf{1} & 1 & 3 & 3 & 4 & 4 \\ 1 & 3 & 3 & 4 & 4 & \mathbf{7} \\ 3 & 3 & 4 & 4 & \mathbf{7} & 7 \\ 3 & 4 & 4 & \mathbf{7} & 7 & 8 \\ 4 & 4 & \mathbf{7} & 7 & 8 & 8 \end{pmatrix}$$

FIGURE 3. The matrices $(\nu(b_{j,k}))_{j,k=1,\ldots,6}$ and $(\nu^*_{j+k-1})_{j,k=1,\ldots,6}$

permutation that produces the minimal value for $\sum_j \nu(b_{j,\sigma_j})$ is indicated in bold typeface. We find that $\nu(\det B_6) = \sum_j \nu(b_{j,\sigma_j}) = -30$.

The claim is proved using induction on $m$ and requires two steps. The first step says that the optimal permutation must restrict to a permutation of $\{2^q-2m+1,\ldots,2m\}$, and the second step specifies its values on this set. The second step establishes the base of the induction as a special case; the base of the induction occurs when $2m = 2^q$, $q \geq 1$.

STEP 1: If $\tau_j \leq 2^q-2m$ for $2^q-2m < j \leq 2m$, then there is a permutation $\tau^*$ such that $\sum_{l=1\ldots 2m}\nu^*_{l+\tau_l-1} < \sum_{l=1\ldots 2m}\nu^*_{l+\tau^*_l-1}$.

*Proof of Step 1.* The proof of this step is inductive on $j$, beginning with $j = 2m$. So suppose first that $\tau_{2m} \le 2^q - 2m$ and define

$$\tau_l^* = \begin{cases} 2^q - 2m + 1 & \text{if } l = 2m \\ \tau_{2m} & \text{if } l = j' \stackrel{\text{def}}{=} \tau^{-1}_{2^q-2m+1} \\ \tau_l & \text{otherwise} \end{cases}$$

so that $\tau^*_{2m} = \tau_{j'} = 2^q - 2m + 1$ and $\tau^*_{j'} = \tau_{2m}$; otherwise, $\tau^*$ and $\tau$ agree. Then,

$$\begin{aligned} \nu^*_{2m+\tau_{2m}-1} + \nu^*_{j'+\tau_{j'}-1} &= \sum_{p>0} \lfloor \frac{2m+\tau_{2m}-1}{2^p} \rfloor + \lfloor \frac{j' + (2^q-2m+1)-1}{2^p} \rfloor \\ &< \sum_{p>0} \lfloor \frac{2^q + j' + \tau_{2m} - 1}{2^p} \rfloor = \sum_{p>0} \lfloor \frac{2^q}{2^p} \rfloor + \lfloor \frac{j' + \tau_{2m} - 1}{2^p} \rfloor \\ &= \sum_{p>0} \lfloor \frac{2m + \tau^*_{2m} - 1}{2^p} \rfloor + \lfloor \frac{j' + \tau^*_{j'} - 1}{2^p} \rfloor = \nu^*_{2m+\tau^*_{2m}-1} + \nu^*_{j'+\tau^*_{j'}-1}. \end{aligned}$$

Here, the first and last equalities are by definition, the strict inequality occurs since both $2m + \tau_{2m} - 1 < 2^q$ and $j' + 2^q - 2m < 2^q$, and the second equality occurs since $j' + \tau_{2m} - 1 \le 2m + (2^q - 2m) - 1 < 2^q$. It follows that $\sum_{l=1\ldots 2m} \nu^*_{l+\tau_l-1} < \sum_{l=1\ldots 2m} \nu^*_{l+\tau^*_l-1}$.

For the inductive step, pick the *largest* $j$ such that $\tau_j \le 2^q - 2m$ and $2^q - 2m < j \le 2m$. Like before, define $\tau'$ by

$$\tau'_l = \begin{cases} 2^q - j + 1 & \text{if } l = j \\ \tau_j & \text{if } l = j' \stackrel{\text{def}}{=} \tau^{-1}_{2^q-j+1} \\ \tau_l & \text{otherwise} \end{cases}$$

so that $\tau'_j = \tau_{j'} = 2^q - j + 1$ and $\tau'_{j'} = \tau_j$. Consider the two possible cases:

*Case 1:* If $j > j'$ then $\sum_{l=1\ldots 2m} \nu^*_{l+\tau_l-1} < \sum_{l=1\ldots 2m} \nu^*_{l+\tau'_l-1}$, since as before,

$$\begin{aligned} \nu^*_{j+\tau_j-1} + \nu^*_{j'+\tau_{j'}-1} &= \sum_{p>0} \lfloor \frac{j+\tau_j-1}{2^p} \rfloor + \lfloor \frac{j' + (2^q - j + 1) - 1}{2^p} \rfloor \\ &< \sum_{p>0} \lfloor \frac{2^q + j' + \tau_j - 1}{2^p} \rfloor = \sum_{p>0} \lfloor \frac{2^q}{2^p} \rfloor + \lfloor \frac{j' + \tau_j - 1}{2^p} \rfloor \\ &= \sum_{p>0} \lfloor \frac{j + \tau'_j - 1}{2^p} \rfloor + \lfloor \frac{j' + \tau'_{j'} - 1}{2^p} \rfloor = \nu^*_{j+\tau'_j-1} + \nu^*_{j'+\tau'_{j'}-1}. \end{aligned} \tag{5}$$

Here, the strict inequality occurs since $j + \tau_j - 1 \le 2m + (2^q - 2m) - 1 < 2^q$ and $j' + (2^q - j + 1) - 1 < j + (2^q - j) = 2^q$, and the second equality occurs since $j' + \tau_j - 1 \le 2m + (2^q - 2m) - 1 < 2^q$.

*Case 2:* If $j < j'$, then the inequality in (5) is weak since $j' + (2^q - j + 1) - 1 > 2^q$. Nevertheless, we recover $\sum_{l=1\ldots 2m} \nu^*_{l+\tau_l-1} \le \sum_{l=1\ldots 2m} \nu^*_{l+\tau'_l-1}$ for the permutation $\tau'$ for which $\tau'_{j'} = \tau_j \le 2^q - 2m$ and $j' > j$. Inductively, then, there is a permutation $\tau^*$ so that

$$\sum_{l=1\ldots 2m} \nu^*_{l+\tau_l-1} \le \sum_{l=1\ldots 2m} \nu^*_{l+\tau'_l-1} < \sum_{l=1\ldots 2m} \nu^*_{l+\tau^*_l-1}$$

and the proof of Step 1 is complete. □

STEP 2: If $\tau_j \neq 2^q - j + 1$ for some $2^q - 2m < j \leq 2m$ then there is a permutation $\sigma$ such that $\sum_{l=1\ldots 2m} \nu^*_{l+\tau_l-1} < \sum_{l=1\ldots 2m} \nu^*_{l+\sigma_l-1}$.

*Proof of Step 2.* After Step 1 we may assume that $2^q - 2m < \tau_j \leq 2m$ for all $2^q - 2m < j \leq 2m$, and we compute

$$\sum_{\substack{p>0 \\ 2^q-2m<j\leq 2m}} \lfloor \frac{j+\tau_j-1}{2^p} \rfloor = \sum_{\substack{p=q \\ 2^q-2m<j\leq 2m}} \lfloor \frac{j+\tau_j-1}{2^p} \rfloor + \sum_{\substack{0<p<q \\ 2^q-2m<j\leq 2m}} \lfloor \frac{j+\tau_j-1}{2^p} \rfloor,$$

since $j + \tau_j - 1 < 2^{q+1}$ for all $j$. The first sum on the right-hand side gives the number of $j$ for which $j + \tau_j - 1 \geq 2^q$, and since

$$\sum_{2^q-2m<j\leq 2m} (j+\tau_j-1) = 2 \cdot [(2^q-2m+1) + \cdots + (2m)] - [2m - (2^q - 2m)]$$
$$= (4m - 2^q)2^q,$$

the second sum is no larger than

$$\sum_{0<p<q} \lfloor \frac{(4m-2^q)2^q}{2^p} \rfloor = (4m-2^q) \cdot \left[2^{q-1} + \cdots + 2\right] = (4m-2^q)(2^q-2).$$

It follows that $\sum_{2^q-2m<j\leq 2m} \nu^*_{j+\tau_j-1}$ is no larger than

$$(4m-2^q) + (4m-2^q)(2^q-2) = (4m-2^q)(2^q-1),$$

with equality possible only if $j + \tau_j - 1 \geq 2^q$ for all $2^q - 2m < j \leq 2m$. But this requires that $\tau_j = 2^q - j + 1$ for all such $j$, and in that case there is equality, since

$$\sum_{\substack{p>0 \\ 2^q-2m<j\leq 2m}} \lfloor \frac{2^q}{2^p} \rfloor = (4m-2^q)(2^{q-1} + \cdots + 1) = (4m-2^q)(2^q-1).$$

This completes the proof of Step 2, so the claim is also proved. □

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