Next week: Tue Wed   Fallback: Wed Thurs

Read 4.1–4.4 tonight (SR)

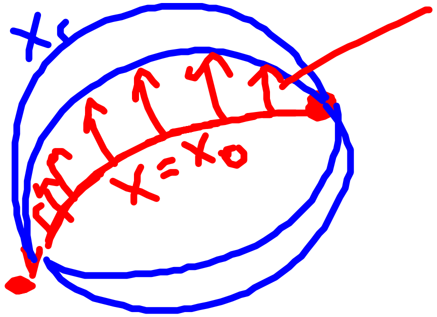
Last time: looking for a condition for a curve $x : [a,b] \to \mathbb{R}^3$ to be an extremum of the functional

$$\Phi(x) = \int_a^b L(x(t), \dot{x}(t))\,dt.$$

We set

$$F(s) = \Phi(x_s) = \int_a^b L(x_s(t), \dot{x}_s(t))\,dt$$

for any variation $\{x_s\}$ of x (recall $x_0 = x$).



$\vec{h} : [a,b] \to \mathbb{R}^3$ variational vector field

We want to find extrema, i.e. curves that satisfy the condition

$$F'(0) = \frac{dF}{ds}\Big|_{s=0} = 0.$$

$$\frac{dF}{ds}\bigg|_{s=0} = \frac{d}{ds}\bigg|_{s=0} \int_a^b L(x_s(t), \dot{x}_s(t))\,dt$$

$$= \cdots = \quad \text{(integration by parts)}$$

$$= \int_a^b \left[\frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} \right] h^k \, dt = 0$$

for any variational vector field $\vec{h} = h^k \hat{e}_k$.

Lemma. $\int_a^b f(x)g(x)\,dx = 0$ for any cont. fn g

$\Longrightarrow f(x) \equiv 0.$

Proof: Arnold, *Mathematical Foundations of Classical Mechanics*.

$$\Longrightarrow \boxed{\frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} = 0, \quad 1 \le k \le 3}$$

Euler-Lagrange Equations

Notes

1) The E-L equations are differential equations and so they give <u>local</u> necessary and sufficient conditions for $x: [a,b] \to \mathbb{R}^3$ to be an extremal

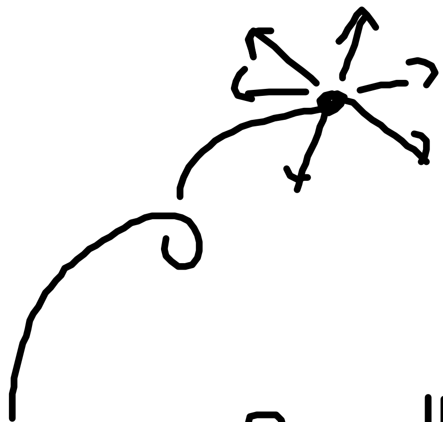
2) Note that the dimensionality (3-dimensions) was irrelevant to our calculations!
So these calculations go through for curves in n-dimensional space $\mathbb{R}^n$

$$\boxed{\begin{gathered} x : [a,b] \to \mathbb{R}^n. \\ \frac{\partial L}{\partial x^k} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^k} = 0 \qquad 1 \le k \le n \end{gathered}}$$

Euler-Lagrange eqns for curves in $\mathbb{R}^n$.

3) The $\dot{x}^k$ variables are actually not time-derivatives of the x^k variables, except along a solution curve of the E-L equations.

In the variational calculus, we actually consider not just all possible positions $\{x^k\}$ but at each point, all possible velocities!



At each point in $\mathbb{R}^3$, there is a 3-dimensional vector space of possible velocities.

So really we're dealing with 6 dimensions: three of position (x^k) and 3 of velocity $(\dot{x}^k)$

For curves in $\mathbb{R}^n$, we're looking at $2n$ dimensions.

Examples

1) Plane geometry.

The arclength of a curve $y = f(t)$, $a \le t \le b$, is

$$S(y) = \int_a^b \sqrt{1 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (*)$$

Thm (Euler) : a curve y minimizes (*)

if and only if it minimizes

$$\Phi(y) = \int_a^b \left[1 + \left(\frac{dy}{dt}\right)^2\right] dt.$$

$$L(y(t), \dot{y}(t)) = 1 + \dot{y}^2$$

$$L(y, y) = 1 + \dot{y}^2$$

E-L: $\dfrac{\partial L}{\partial x^k} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}^k} = 0 \quad 1 \le k \le n$

$$\left.\begin{aligned} x^1 &= y \\ \dot{x}^1 &= \dot{y} \end{aligned}\right\} \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

$$= 0 - \frac{d}{dt}(2\dot{y}) = 0$$

$$-2\ddot{y} = 0 \iff \ddot{y} = 0.$$

So a curve in the plane is an extremal for $L = 1 + \dot{y}^2$ iff $\ddot{y} = 0$.

$$\ddot{y} = 0 \implies \dot{y} = m \quad (\text{const})$$

$$y = mt + b \quad (b \text{ const})$$

$\therefore$ extremals of L are straight lines.

Example 2 : conservative force field $\vec{F} = -\nabla \Phi$

where Φ = potential function. $\Phi(x,y,z)$

$$L = T - V \qquad (KE - PE)$$

$$= \frac{1}{2} m v^2 - \Phi \qquad v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \Phi$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$-\frac{\partial}{\partial x}\Phi - \frac{d}{dt}(m\dot{x}) = 0$$

$$-\frac{\partial}{\partial x}\Phi - \frac{d}{dt} m\dot{x} = 0$$

$$-\frac{\partial}{\partial x}\Phi - m\ddot{x} = 0$$

$$\Rightarrow \quad m\ddot{x} = -\frac{\partial}{\partial x}\Phi .$$

Similar calculations with the other two E-L equations give:

$$m\ddot{y} = -\frac{\partial}{\partial y}\Phi ,$$

$$m\ddot{z} = -\frac{\partial}{\partial z}\Phi .$$

Thus,

$$m\ddot{x} = -\frac{\partial}{\partial x}\Phi$$

$$m\ddot{y} = -\frac{\partial}{\partial y}\Phi$$

$$m\ddot{z} = -\frac{\partial}{\partial z}\Phi$$

so

$$m\ddot{x}\hat{\imath} + m\ddot{y}\hat{\jmath} + m\ddot{z}\hat{k} = -\left(\frac{\partial}{\partial x}\Phi\hat{\imath} + \frac{\partial}{\partial y}\Phi\hat{\jmath} + \frac{\partial}{\partial z}\Phi\hat{k}\right)$$

$$m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}) = -\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\Phi$$

$$\Rightarrow \quad m\vec{a} = -\nabla\Phi.$$

So the E-L equations for $L = T - V$
are equivalent to the vector equation

$$-\nabla \Phi = m\vec{a};$$

and now, recalling that $\vec{F} = -\nabla \Phi$, we obtain

$$\vec{F} = m\vec{a}.$$

That is, we have recovered Newton's 2nd Law
from the E-L equations.

Summary: these two examples show how the variational calculus provides a unified method for handling a wide variety of optimal path problems.

In geometry, solutions to the E-L equations are called geodesics. In classical mechanics, they are called paths of least action.