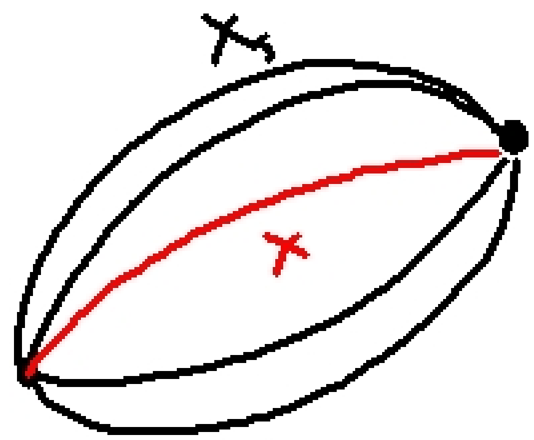
Variational Calculus, cont'd

Last time: given a smooth curve

$$X : [a,b] \to \mathbb{R}^3$$



a variation of X is a 1-parameter family of curves $X_s : [a,b] \to \mathbb{R}^3$ for $s \in (-\varepsilon, \varepsilon)$

and

$$X_0(t) = X(t) \quad \forall t \in [a,b] \qquad (\text{i.e. } X_0 = X)$$

$$\left.\begin{array}{l} X_s(a) = X(a) \\ X_s(b) = X(b) \end{array}\right\} \forall s \in (-\varepsilon, \varepsilon)$$

We defined a <u>functional</u> Φ on the space of curves by

$$\Phi(x_s) = \int_a^b L(x_s(t), \dot{x}_s(t), t)\, dt \qquad \begin{pmatrix} \text{action} \\ \text{integral} \end{pmatrix}$$

where L is called the Lagrangian (cost function).

Special case: $L = L(x(t), \dot{x}(t))$

$$\Phi(x) = \int_a^b L(x(t), \dot{x}(t))\, dt.$$

Examples:

1) Plane geometry $\quad \vec{X}(t) = (x(t), y(t))$

$$L(\vec{X}(t), \dot{\vec{X}}(t)) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Special case where $y = f(x)$

$$L(\vec{x}, \dot{\vec{x}}) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

2) Classical mechanics:

$$L = T - V$$

where T = kinetic energy

V = potential energy.

Given $\Phi(x_s) = \int_a^b L(x_s(t), \dot{x}_s(t))\,dt$, "take the derivative and set it to 0" by defining

$$F(s) = \Phi(x_s) = \int_a^b L(x_s(t), \dot{x}_s(t))\,dt$$

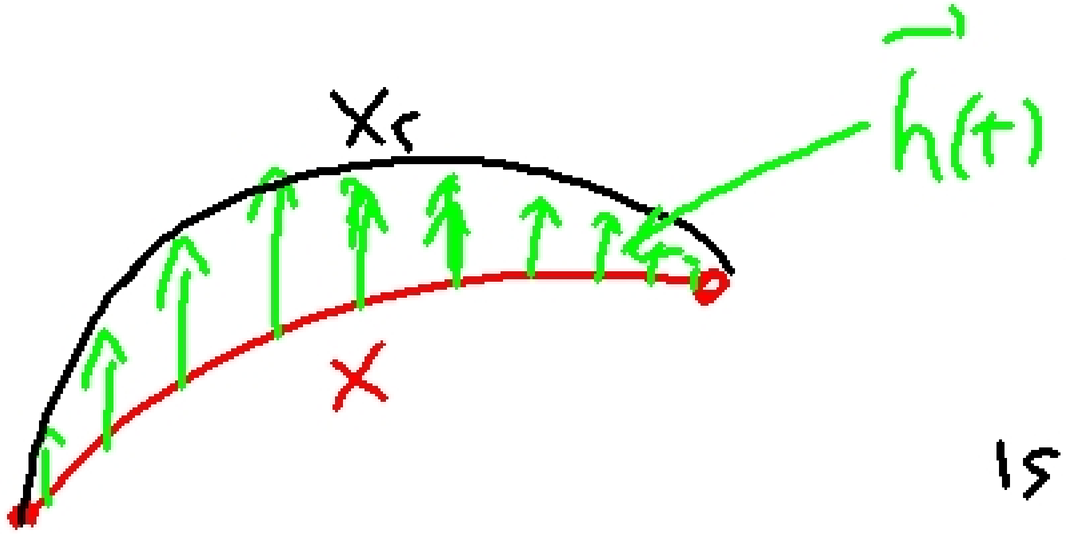
and take $F'(0) = \left.\frac{dF}{ds}\right|_{s=0} = 0$.

N.B. Even in one-variable calculus, solutions of this equation are not guaranteed to be <u>minima</u>! But every local minimum is a solution.

<u>def.</u> A curve $x : [a,b] \to \mathbb{R}^3$ is an <u>extremum</u> <u>for L</u> (or <u>L-critical</u>) if

$$F'(0) = 0 \qquad (x_0 = x)$$

for all variations $\{x_s\}$ of x.



$X_s(t) = X(t) + s\,h(t)$

is a variation of X when

$h : [a,b] \to \mathbb{R}^3$ vector field, smooth

$h(a) = 0$

$h(b) = 0$

h is called a <u>variational vector field</u>.

In terms of components,

$$h(t) = h^1 \hat{e}_1 + h^2 \hat{e}_2 + h^3 \hat{e}_3$$

where

$$\left.\begin{aligned} \hat{e}_1 &= (1,0,0) \\ \hat{e}_2 &= (0,1,0) \\ \hat{e}_3 &= (0,0,1) \end{aligned}\right\} \text{std basis for } \mathbb{R}^3$$

$$h(t) = \sum_{k=1}^{3} h^k \hat{e}_k = h^k \hat{e}_k \quad \left(\begin{array}{c}\text{Einstein}\\ \text{summation}\\ \text{convention}\end{array}\right)$$

$$X(t) = x^k(t)\,\hat{e}_k$$

$$\dot{X}(t) = \dot{x}^k(t)\,\hat{e}_k$$

Now: we'll calculate $F'(0) = \left.\frac{dF}{ds}\right|_{s=0} = 0.$

$$F(s) = \int_a^b L(x_s(t), \dot{x}_s(t))\,dt$$

$$\frac{dF}{ds}\Big|_{s=0} = \frac{d}{ds}\Big|_{s=0} \int_a^b L(x_s(t), \dot{x}_s(t))\,dt$$

$$(x_s(t) = x(t) + s h(t))$$

$$= \frac{d}{ds}\Big|_{s=0} \int_a^b L(x(t)+sh(t), \dot{x}(t)+s\dot{h}(t))\,dt$$

$$= \int_a^b \frac{\partial}{\partial s} L(x(t)+sh(t), \dot{x}(t)+s\dot{h}(t))\,dt$$

$$= \int_a^b \frac{\partial}{\partial s} L\left(x(t) + sh(t),\ \dot{x}(t) + s\dot{h}(t) \right) dt$$

$$= \int_a^b \left(\sum_{k=1}^{3} \frac{\partial L}{\partial x^k} \cdot \frac{\partial x^k}{\partial s} + \frac{\partial L}{\partial \dot{x}^k} \cdot \frac{\partial \dot{x}^k}{\partial s} \right) dt$$

$$= \int_a^b \left(\frac{\partial L}{\partial x^k} \frac{\partial x^k}{\partial s} + \frac{\partial L}{\partial \dot{x}^k} \frac{\partial \dot{x}^k}{\partial s} \right) dt \qquad \left(\begin{array}{c} \text{Chain} \\ \text{Rule} \end{array} \right)$$

$$= \int_a^b \left(\frac{\partial L}{\partial x^k} h^k + \frac{\partial L}{\partial \dot{x}^k} \dot{h}^k \right) dt$$

$$= \int_a^b \frac{\partial L}{\partial x^k} h^k \, dt + \underbrace{\int_a^b \frac{\partial L}{\partial \dot{x}^k} \dot{h}^k \, dt} \qquad (*)$$

$$\int_a^b \frac{\partial L}{\partial \dot{x}^k} \dot{h}^k \, dt = \frac{\partial L}{\partial \dot{x}^k} h^k \Big|_a^b - \int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) h^k \, dt$$

$$u = \frac{\partial L}{\partial \dot{x}^k} \qquad dv = \dot{h}^k \, dt$$

$$du = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) dt \qquad v = h^k$$

$$= \frac{\partial L}{\partial \dot{x}^k} \cancel{h^k(b)}^{\,0} - \frac{\partial L}{\partial \dot{x}^k} \cancel{h^k(a)}^{\,0} - \int_a^b \sim$$

$$= -\int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) h^k \, dt$$

$$(*) = \int_a^b \underbrace{\left[\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right]}_{\text{identically } 0} h^k \, dt = 0$$

identically 0 if this is true for any smooth $\vec{h}$ vanishing at endpoints

$$\boxed{\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) = 0, \qquad 1 \le k \le 3}$$

Euler-Lagrange equations