5/12/05

Today: - escape velocity in Schwarzschild spacetime

- orbits near a neutron star

- black holes

Tomorrow: last class!

- class pictures
- black holes
- TCE review, end.

Escape velocity

Newtonian model

$$V(r) = -\frac{GM}{r}$$

$$T = \frac{1}{2}mv^2$$

Launch projectile from radius $r = R$ away from center of mass at minimum velocity v so that projectile reaches infinity.

At infinity,

$$mV(r) = -\frac{GMm}{r}$$

velocity $v = 0$ (projectile at rest)

$$E = T + V$$

$$= \frac{1}{2}mv^2 - \frac{GMm}{r} = 0 \quad \text{at infinity}$$

Conservation of energy $\Rightarrow$ $E = 0$ at all points

$$\Rightarrow \frac{1}{2}mv^2 - \frac{GMm}{r} = 0$$ for all points on orbit

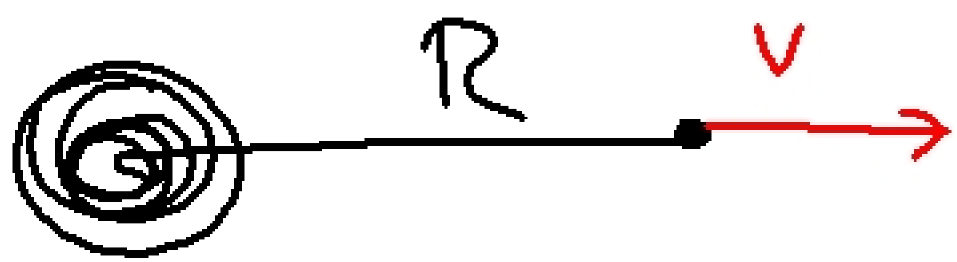
$$\Rightarrow \frac{1}{2}mv^2 = \frac{GMm}{r}$$

At start, where $r = R$

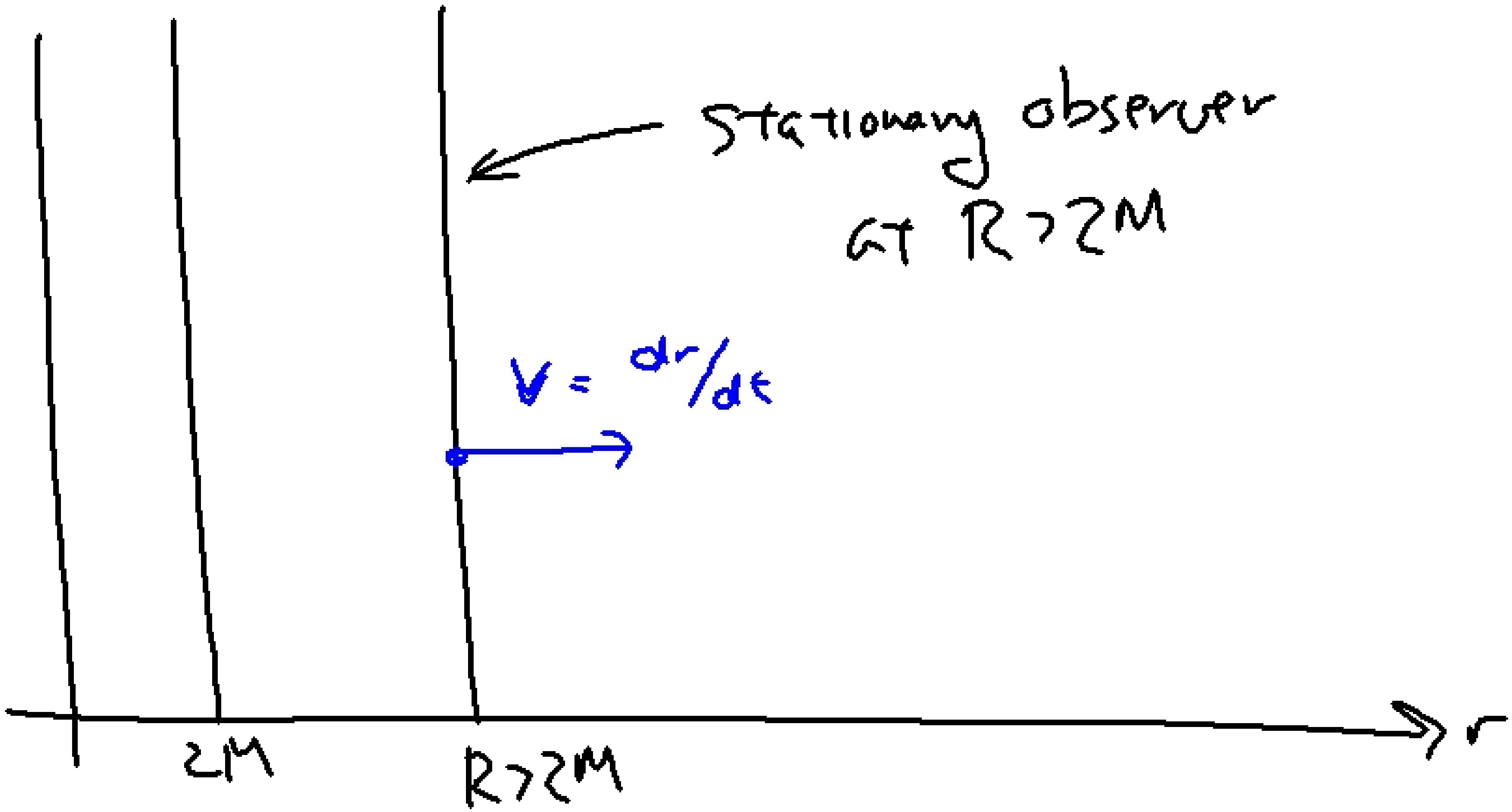
$$\frac{1}{2}mv^2 = \frac{GMm}{R}$$

$$\frac{1}{2}v^2 = \frac{GM}{R}$$

$$V = \sqrt{\frac{2GM}{R}}$$ escape velocity at $r = R$



Schwarzschild version



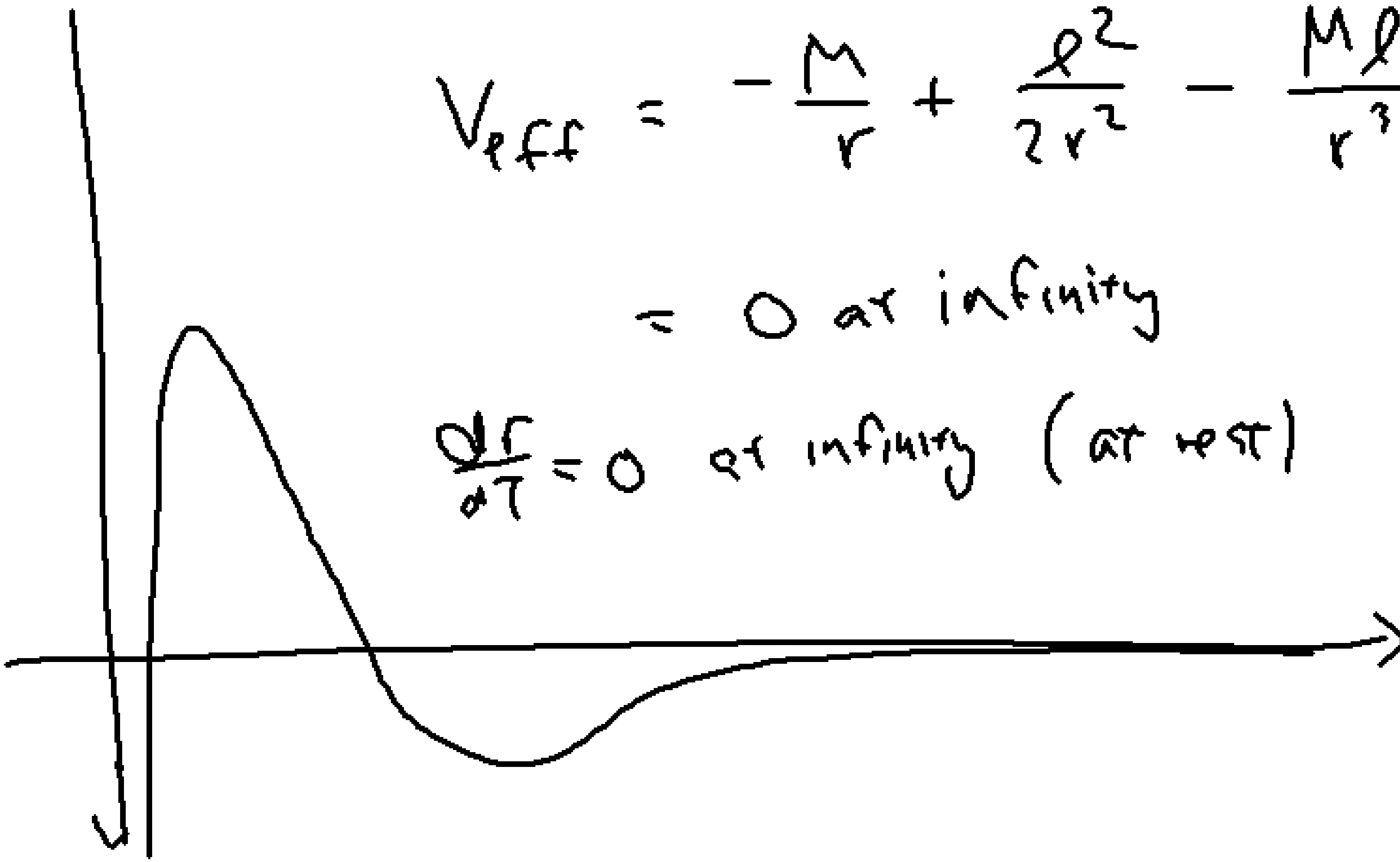
- Projectile follows radial geodesic.

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff.}$$

$$V_{eff} = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

$= 0$ at infinity

$\frac{dr}{d\tau} = 0$ at infinity (at rest)



$\mathcal{E} = 0$ at infinity — constant on geodesics

1) $\Rightarrow \mathcal{E} = 0$ at all points on the orbit.

2) radial geodesic $\Rightarrow \ell = 0$ at all points

$$\boxed{\mathcal{E} = \frac{e^2 - 1}{2} \Rightarrow e = 1 \text{ at all points.}}$$

(In other words projectile must be launched at high enough velocity so that $e \geq 1$.)

$$\frac{e^2-1}{2} = \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

$$= \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right) - 1\right]$$

$$0 = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\cdot 1 - 1\right]$$

$$= \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(-\frac{2M}{r}\right)$$

$$0 = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r}.$$

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{M}{r}$$

so at $r = R$,

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{M}{R}$$

$$\frac{dr}{d\tau} = \pm\sqrt{\frac{2M}{R}}$$

outward motion: wrt τ

$$\frac{dr}{d\tau} = \sqrt{\frac{2M}{R}}.$$

4-velocity of projectile at launch

$$\bar{u}^\alpha = (u^t, u^r, u^\theta, u^\varphi)$$

$$= \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right)$$

$$= \left(\frac{dt}{d\tau}, \sqrt{\frac{2M}{R}}, 0, 0\right)$$

What is this?

p. 193 of Hartle:

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$$

At $r = R$,

$$e = \left(1 - \frac{2M}{R}\right)\frac{dt}{d\tau} = 1$$

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{R}\right)^{-1}$$ at launch.

$$u^\alpha = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right)$$

$$u^\alpha = \left(\left(1-\frac{2M}{R}\right)^{-1}, \sqrt{\frac{2M}{R}}, 0, 0\right) \text{ at launch}$$

4-momentum at launch:

$$\bar{p} = m\bar{u}$$

Energy observed by an observer with 4-velocity $\bar{u}_{obs}$

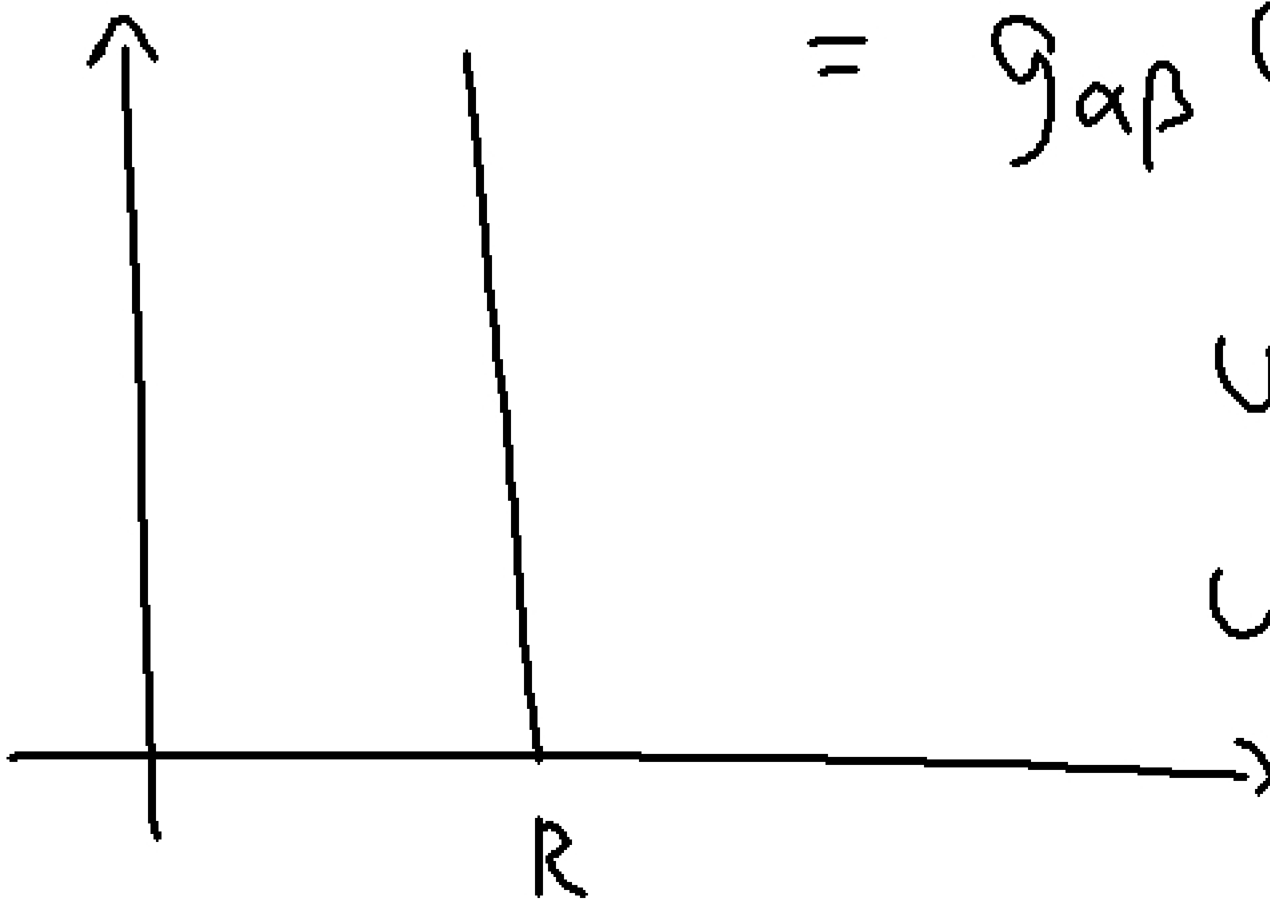
is $$E = -\bar{p} \cdot \bar{u}_{obs}$$ (5.87 in Hartle)

$$E = -\bar{p} \cdot \bar{u}_{obs}$$

$\bar{u}_{obs} = ?$

$$u_{obs} \cdot u_{obs} = -1$$

$$= g_{\alpha\beta}\, u^{\alpha}_{obs} u^{\beta}_{obs}$$



$u^{r}_{obs} = 0, \; u^{\theta}_{obs} = 0, \; u^{\varphi}_{obs} = 0$

$u^{t} = ?$

$$\bar{u}_{obs} \cdot \bar{u}_{obs} = g_{tt}(u^t_{obs})^2 = -1$$

$$= -\left(1 - \frac{2M}{r}\right)(u^t_{obs})^2 = -1$$

$$(u^t_{obs})^2 = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$u^t_{obs} = \left(1 - \frac{2M}{r}\right)^{-1/2}.$$

$$\bar{u}_{obs}^{\alpha} = \left(\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0 \right)$$

$$\bar{u}^{\alpha} = \left(\left(1-\frac{2M}{R}\right)^{-1}, \left(\frac{2M}{R}\right)^{1/2}, 0, 0 \right) \text{ at launch}$$

$$E = -\bar{p} \cdot u_{obs}$$

$$= -m\,\bar{u} \cdot \bar{u}_{obs}$$

$$= -m\, g_{tt}\, u^{t}\, u^{t}_{obs} = -m\left(-\left(1-\frac{2M}{R}\right)\right)\left(1-\frac{2M}{R}\right)^{-1/2}\left(1-\frac{2M}{R}\right)^{-1}$$

$$= m\left(1-\frac{2m}{R}\right)^{-1/2}$$

$$E = m\left(1 - \frac{2M}{R}\right)^{-1/2} = \frac{m}{\sqrt{1 - \frac{2M}{R}}}$$

$$E = \frac{m}{\sqrt{1 - v^2}}$$

v = 3-velocity $\left(\frac{dr}{d\tau}\right)$

$$\Rightarrow v^2 = \frac{2M}{R}$$

$$\boxed{v = \sqrt{\frac{2M}{R}}}$$

For a collapsing shell of matter,

as it collapses to $R < 2M$,

at the point where $R = 2M$

$$V = \sqrt{\frac{2M}{2M}} = 1$$