3/3/05

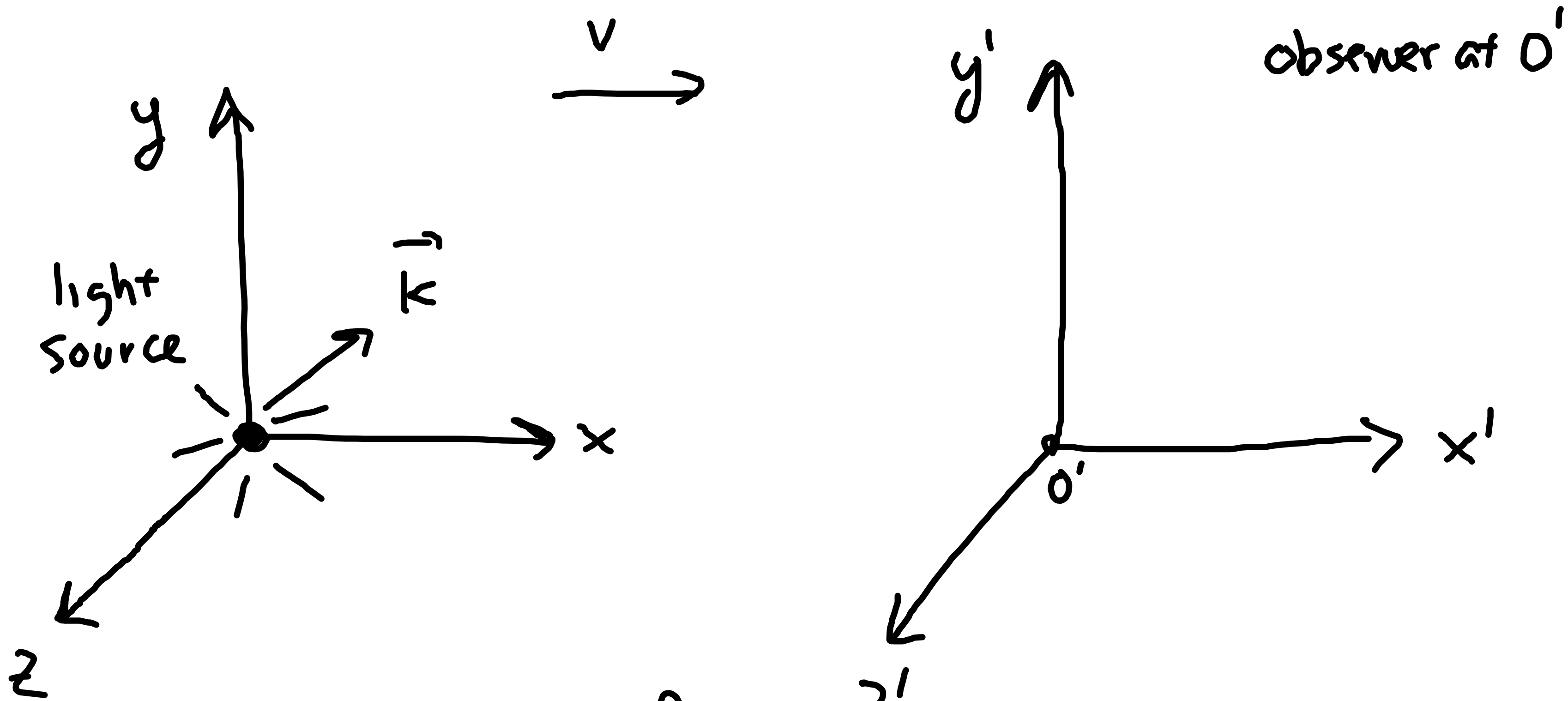
WPR1 Thursday next week

- closed book, closed notes
- calc allowed
- no laptop
- do prob. 6, 7, 8, 12 in ch. 5

Recall: a photon of frequency ω has an associated wave 4-vector

$$\bar{k} = (\omega, \vec{k}), \quad |\vec{k}| = \omega$$

Doppler shift: imagine a source emitting light, moving with velocity v relative to an observer.



In the source rest frame,

$$\bar{k} = (k^t, k^x, k^y, k^z) = (\omega, \vec{k})$$

$$k^t = \omega, \quad \vec{k} = (k^x, k^y, k^z)$$

Observer's frame: t', x', y', z'

unprimed frame moves along x' axis with velocity v

$\iff$ primed frame moves along x-axis with velocity $-v$. $\bar{k} = (\omega', \vec{k}')$ in primed frame

$$\omega' = k^{t'} = \gamma(k^t - (-v)k^x) = \gamma(\omega + vk^x)$$

$$k^{x'} = \gamma(k^x - (-v)k^t) = \gamma(k^x + v\omega)$$

$$k^{y'} = k^y$$

$$k^{z'} = k^z.$$

$$\omega' = \gamma(\omega + v k^x)$$

Inverse transform:

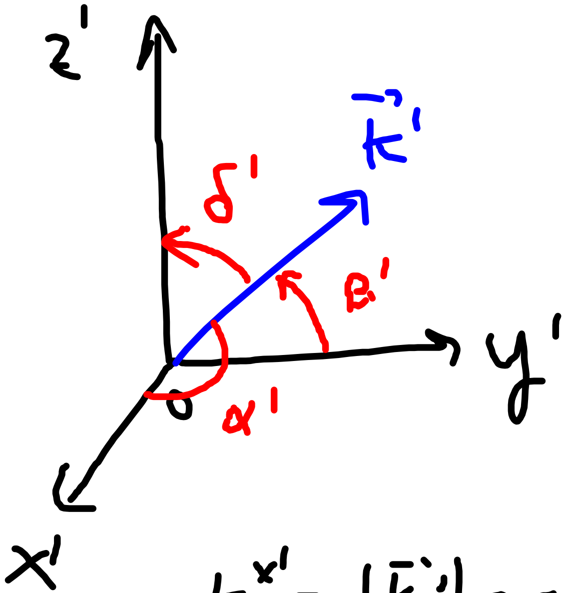
$$\omega = \gamma(\omega' - v k^{x'})$$

What is $k^{x'}$?

In observer's (primed) frame,

the wave 3-vector $\vec{k}'$ has components

$$\vec{k}' = (k^{x'}, k^{y'}, k^{z'})$$



Linear algebra fact:

$$\vec{k}' = (|\vec{k}'| \cos\alpha')\hat{e}_{x'} + (|\vec{k}'| \cos\beta')\hat{e}_{y'}$$
$$+ (|\vec{k}'| \cos\delta')\hat{e}_{z'}$$

$$k^{x'} = |\vec{k}'| \cos\alpha', \quad k^{y'} = |\vec{k}'| \cos\beta',$$

$$k^{z'} = |\vec{k}'| \cos\delta'.$$

$$|\vec{k}| = \omega, \quad |\vec{k}'| = \omega'$$

$$k^{x'} = |\vec{k}'| \cos\alpha'$$

$$k^{x'} = \omega' \cos\alpha'$$

$$\left.\begin{aligned} \omega &= \gamma(\omega' - v k^{x'}) \\ &= \gamma(\omega' - v\omega'\cos\alpha') \\ &= \gamma\omega'(1 - v\cos\alpha') \end{aligned}\right\} \Rightarrow \omega' = \frac{\omega}{\gamma} \cdot \frac{1}{1 - v\cos\alpha'}$$

$$= \frac{\omega\sqrt{1-v^2}}{1 - v\cos\alpha'}$$

$$\boxed{\omega' = \frac{\omega\sqrt{1-v^2}}{1-v\cos\alpha'}}$$ Doppler shift

$$= \omega\sqrt{1-v^2}\left(1+v\cos\alpha' + (v\cos\alpha')^2 + \cdots\right)$$

For $v \ll 1$, this is

$$\omega' \approx \omega\,(1+v\cos\alpha').$$

For $v \ll 1$,

$$\omega' \approx \omega(1 + v\cos\alpha')$$

$\underline{\alpha' = 0}$: photon emitted in same direction that source is moving : $\omega' \approx \omega(1+v)$

$$\Delta\omega' \approx v\omega \quad \text{blue shifted}$$

$\underline{\alpha' = \pi}$: photon emitted in opposite direction

$$\Delta\omega' = -v\omega \quad \text{red shifted.}$$

$\underline{\alpha' = \frac{\pi}{2}}$: photon is emitted at right angle to source.

$$\omega' = \frac{\omega\sqrt{1-v^2}}{1 - v\cos\alpha'} \qquad \alpha' = \frac{\pi}{2}$$

$$\omega' = \omega\sqrt{1-v^2} < \omega$$

time dilation affects ω!

$k^x = \omega \cos\alpha$

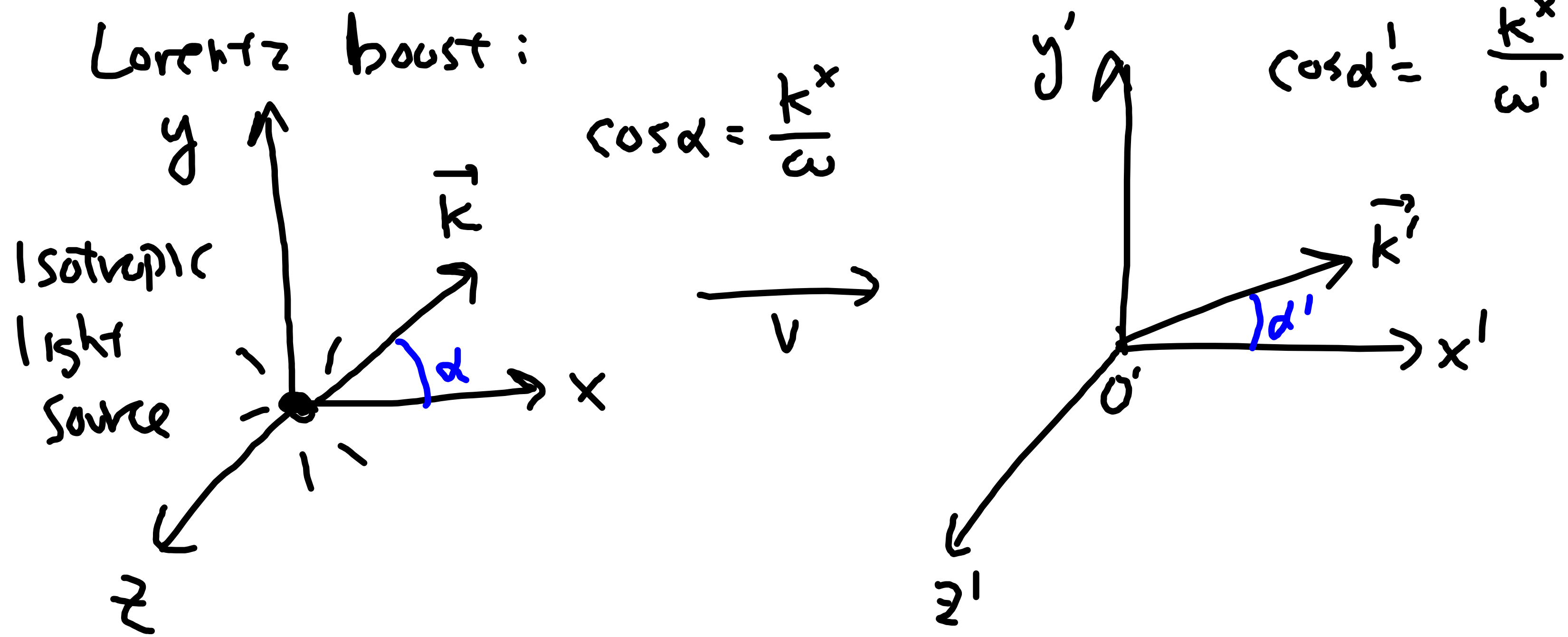
Relativistic Beaming

Spatial (3-momentum) of photon is affected by Lorentz boost:

$\cos\alpha = \frac{k^x}{\omega}$

$\cos\alpha' = \frac{k^{x'}}{\omega'}$

Isotropic light source



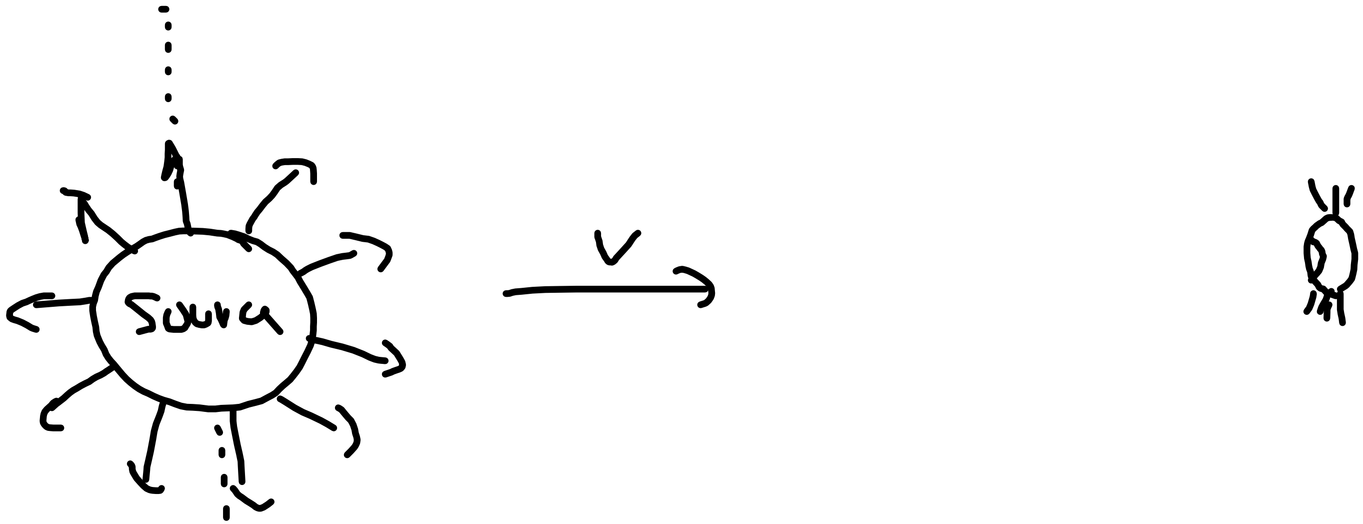
$$\omega' = k^{t'} = \gamma(k^t + v k^x) = \gamma(\omega + v k^x)$$

$$k^{x'} = \gamma(k^x + v\omega)$$

$$\cos\alpha' = \frac{k^{x'}}{\omega'} = \frac{\gamma(k^x + v\omega)}{\gamma(\omega + v k^x)}$$

$$= \frac{k^x + v\omega}{v k^x + \omega} \cdot \frac{\frac{1}{\omega}}{\frac{1}{\omega}} = \frac{\frac{k^x}{\omega} + v}{v\frac{k^x}{\omega} + 1} = \frac{\cos\alpha + v}{1 + v\cos\alpha}$$

$$\cos\alpha' = \frac{\cos\alpha + V}{1 + V\cos\alpha}$$



For $|\alpha| < \frac{\pi}{2}$, $|\alpha| \to \frac{\pi}{2}$

$$\cos\alpha' \longrightarrow V$$

$\cos\alpha' \rightarrow v$

If v is close to 1, α' is close to 0