

Optimal Enforcement and Adequate Penalties

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Abstract

When inspection is used to enforce a rule in a context such as arms control, auditing, or food and drug inspection, there is a conflict between inspector and inspectee that can be modeled as a non-zero-sum strategic-form game. Suppose that the inspectee may violate at any subset of n non-identical “sites”, complying at the rest. The inspector chooses $k < n$ sites at which to assess the inspectee’s behavior. We assume that inspections are perfect in that a violation at an inspected site is always detected, and false alarms never occur, and say that the violation penalty is *adequate* if the inspector can select an inspection strategy that rationally induces the inspectee to comply at all sites. We determine these optimal inspection strategies in important classes of inspection games, thereby identifying the minimum adequate penalty, or *threshold*. We then study how the threshold is related to other parameters, principally the amount of inspection (represented by k), which is an important issue for treaty designers and policy-makers.

1 Introduction

Game theory was first applied to arms control during the USA-USSR arms-reduction negotiations of the mid-1960s, the era of *détente*. The United States Arms Control and Disarmament Agency, which was responsible for drafting the US position, believed that game theory could provide valuable insights into the consequences of treaty provisions, and assembled some prominent game theorists to address the strategic issues raised by arms control. The series of reports that resulted (several reprinted in Aumann and Maschler [1995]) is now regarded as a landmark in the history of game theory.

With the entry into force of the Non-Proliferation Treaty in 1970, the International Atomic Energy Agency (IAEA) began regular inspections of nuclear-energy facilities in many countries, the “non-nuclear weapon states.” Supporting game-theoretic studies focused on whether the IAEA’s decision rules for inspectors were equilibrium strategies in any sense. The scope for game-theoretic analysis expanded as subsequent treaties allowed for more inspections under different rules. (See Avenhaus and Canty [1996] for details.) Avenhaus et al. [1996] survey the general class of inspection games, and Avenhaus et al. [2002] connect inspection in arms control, environmental control, and auditing, and investigate the relation of equilibrium inspection strategies to standard statistical procedures.

Our objective is to use game theory to analyze a class of inspection games defined by Kilgour [1992]. The (n, k) inspection games were introduced to study the implications of a trend to include widely different facilities within a single arms-control agreement. For example, under the 1990 Treaty on Conventional Forces in Europe, thousands of sites within the ATTU (Atlantic To The Urals) zone were subject to inspection, though only between 10% and 20% could be inspected in any year. The military characteristics of the inspectable sites varied widely, so violations at some sites would be much more threatening than violations at others. Clearly, the subset of sites to be inspected should be chosen randomly but not uniformly; if an (n, k) game models this problem, then an optimal strategy in the game would be the best subset for inspection.

The (n, k) games are non-zerosum games with two players, called Inspector and Inspectee. The fundamental parameters are n , the number of sites where Inspectee can violate, and k , the number of sites that Inspector may inspect. In addition to Kilgour [1992], other work establishing the importance of (n, k) inspection games includes Kilgour [1990], Kilgour and Brams [1992], and Brams and Kilgour [1988, Ch. 8].

In all (n, k) games, Inspectee has an obligation to comply at every one of the n (in general, non-identical) sites, but is motivated to violate at one or more sites. Inspector attempts to deter violations, or at least minimize them. In a play of the game, Inspectee

chooses to comply or violate at each site and, simultaneously, Inspector selects a subset of sites at which Inspectee’s compliance is verified. Inspection is assumed to be perfect in that (1) detection is certain if a violation site is inspected and (2) false alarms never occur. Of course, (n, k) games are models not only for multi-site arms-control treaties, but also for enforcement problems such as auditing, food and drug regulation, weights and measures verification, and environmental management.

Kilgour [1992] discussed the $(2,1)$ inspection game in detail, showing that if the penalty for a detected violation is high enough, Inspector has a mixed strategy—a probability distribution over the two inspectable sites—that induces Inspectee to comply for certain. He also solved special cases of the $(3,1)$ and $(3,2)$ games, and suggested some features of solutions of the general (n, k) game. Here, important special cases of the (n, k) inspection game are solved in general, and threshold penalties are estimated in a broader class.

2 The (n, k) game and the $(2, 1)$ game

We summarize and generalize the definitions of the (n, k) game originated by Kilgour [1992]. The fundamental parameters, n and k , are positive integers with $k < n$. The sites are numbered $1, 2, \dots, n$. Inspectee selects any subset of the sites (where violations will take place), while Inspector selects for inspection any subset containing up to k sites.

The players’ (von Neumann-Morgenstern) utilities at the outcomes of the (n, k) game are determined by the following parameters:

- v_i is Inspectee’s benefit for an undetected violation at site i ,
- $-w_i$ is Inspector’s cost should there be an undetected violation at site i ,
- $-P$ is the penalty to Inspectee if a violation site is inspected,
- F is the bonus to Inspector if a violation site is inspected,
- e is the “clean inspection” bonus to Inspectee if no violation site is inspected.

These parameters are in units of utility — Inspectee’s for P , e , and the v_i ’s, and Inspector’s for F and the w_i ’s. We assume that the *site value parameters* satisfy $v_i > 0$ and $w_i > 0$ for $i = 1, \dots, n$. Also, we require that $P > 0$, $e > -P$, and $F > -\min\{w_i : i = 1, \dots, n\}$ to guarantee that detection at any site leaves Inspectee worse off than at the status quo outcome (where there is no violation and, therefore, no detection) and that Inspector prefers to detect a violation rather than overlook it. Note that Inspector may prefer a detected

violation to the status quo ($F > 0$, perhaps reflecting that Inspector likes to demonstrate that the system works) or not ($F < 0$, perhaps reflecting that Inspector wants the system to deter violations). Of course, we assume that each party knows the values of all parameters, and thus that it knows its own and its opponent's utilities at every outcome of the game.

Let S denote a pure strategy for Inspectee—that is, a subset of the n sites—and let T denote a pure strategy for Inspector—a subset containing k or fewer sites. Then the players' utilities, $V_e^r(S, T)$ for Inspectee and $V_o^r(S, T)$ for Inspector, satisfy

$$V_e^r(S, T) = \begin{cases} e + \sum_{j \in S} v_j, & \text{if } S \cap T = \emptyset \\ -P, & \text{if } S \cap T \neq \emptyset. \end{cases} \quad (1)$$

$$V_o^r(S, T) = \begin{cases} -\sum_{j \in S} w_j, & \text{if } S \cap T = \emptyset \\ F, & \text{if } S \cap T \neq \emptyset. \end{cases} \quad (2)$$

In (1) and (2), we adopt the convention that a summation over an empty set equals 0.

Note that the consequences of detection, P and F , are determined by the underlying treaty and not decided by Inspector or Inspectee. Of course, these parameters reflect the utilities assigned by the parties, but the treaty designer can change them by altering the consequences of detection. Our objective is to assist treaty designers and policy makers by explicating the behavioral incentives implied by particular rewards and penalties, and showing how these incentives depend on n , k , and the pattern of site value parameters.

It is easy to show that if T' is a strategy of Inspector satisfying $|T'| < k$, then T' is (weakly) dominated by any strategy $T \supset T'$ such that $|T| = k$. Ignoring such dominated strategies, we model Inspector as having $N = \binom{n}{k}$ pure strategies. (Crucial to this argument is the assumption, implicit in (2), that Inspector's costs are sunk: Inspector cannot reduce costs by inspecting fewer sites. See Kilgour and Brams [1992] for further discussion.)

Our assumption that any Inspector strategy contains k sites implies that an Inspectee strategy S' satisfying $|S'| > n - k$ is (weakly) dominated by any strategy $S \subset S'$ such that $|S| = n - k$. Again ignoring these dominated strategies, we model Inspectee as having $M = \sum_{i=0}^{n-k} \binom{n}{i}$ pure strategies.

When written in strategic (normal) form, the (n, k) game is therefore an $M \times N$ bimatrix. To see why this game cannot have a pure-strategy Nash equilibrium, suppose first that (S, T) is an equilibrium with $S \neq \emptyset$. By (1), maximization of Inspectee's utility implies that $S \cap T = \emptyset$ whereas, by (2), maximization of Inspector's utility implies $S \cap T \neq \emptyset$. Clearly these conditions cannot be true simultaneously. As well, (\emptyset, T) cannot be a Nash

equilibrium because, since $|T| = k$, any non-empty S' such that $S' \subseteq T^c$ must satisfy $V_e^r(S', T) > V_e^r(\emptyset, T)$, by (1). (Here, T^c denotes the complement of T .) Thus, the (n, k) game has no pure-strategy equilibria.

Mixed strategies in the (n, k) game are probability distributions over subsets of the sites. Although there is never a pure-strategy equilibrium, there can be an equilibrium where one player's strategy is pure and the other's is mixed. Specifically, Inspector may be able to choose a mixed strategy such that the expected value of (1) is non-positive for any $S \neq \emptyset$. If so, the choice of such a strategy for Inspector, along with $S = \emptyset$ for inspectee, is a Nash equilibrium called a *legal equilibrium*. At any legal equilibrium, Inspector must choose a non-degenerate probability distribution over two or more inspection sets. It can be verified readily that at any *illegal equilibrium*, i.e. any equilibrium that is not legal, neither player chooses a pure strategy.

For example, the $(2, 1)$ game has the following 3×2 strategic form, with Inspectee as row chooser and Inspector as column chooser:

	Search 1	Search 2
Comply	$e, 0$	$e, 0$
Violate 1	$-P, F$	$v_1 + e, -w_1$
Violate 2	$v_2 + e, -w_2$	$-P, F$

Table 1. The $(2,1)$ game

Note that Inspector decides only which of the two sites to inspect, whereas Inspectee decides whether to comply or violate and, if the latter, where to violate.

It is convenient to transform the game of Table 1 to a form with fewer parameters. Let $D = P + e$ and note that, by assumption, $D > 0$. (Thus D is the opportunity cost to Inspectee of a detected violation, which we interpret as the effective penalty.) Adding $(P, -F)$ to every cell in the bimatrix of Table 1 transforms it to the following game:

	Search 1	Search 2
Comply	$D, -F$	$D, -F$
Violate 1	$0, 0$	$v_1 + D, -(F + w_1)$
Violate 2	$v_2 + D, -(F + w_2)$	$0, 0$

Table 2. The transformed $(2,1)$ game

Of course, the games of Tables 1 and 2 are strategically equivalent; in particular, their Nash equilibria are identical.

For the (2,1) game, represent a mixed strategy for Inspectee as a probability 3-vector $\beta = (\beta_0, \beta_1, \beta_2)$, and a mixed strategy for Inspector as a probability 2-vector $\alpha = (\alpha_1, \alpha_2)$. By the Nash Theorem, there is always at least one Nash equilibrium. All generic equilibria, denoted (β^*, α^*) , are given by the next theorem.

Theorem 1. [Kilgour, 1992] Let $D_{crit} = \sqrt{v_1 v_2}$. If $0 < D < D_{crit}$, the unique Nash equilibrium of the (2, 1) game is (β^*, α^*) where

$$\beta_0^* = 0, \beta_1^* = \frac{w_2 + F}{w_1 + w_2 + 2F}, \beta_2^* = 1 - \beta_1^*; \alpha_1^* = \frac{v_1 + D}{v_1 + v_2 + 2D}, \alpha_2^* = 1 - \alpha_1^*.$$

If $D_{crit} < D$, (β^*, α^*) is a Nash equilibrium of the (2, 1) game iff $\beta^* = (1, 0, 0)$ and α^* satisfies

$$\frac{v_1}{v_1 + D} \leq \alpha_1^* \leq \frac{D}{v_2 + D}, \alpha_2^* = 1 - \alpha_1^*.$$

In particular, if $D < D_{crit}$ the game has a unique equilibrium, which is illegal, and if $D > D_{crit}$ it has only legal equilibria.

Thus, in the (2,1) game, the value of D determines whether the Nash equilibria are legal or illegal; at the threshold, when $D = D_{crit}$, legal and illegal equilibria co-exist. It can be shown that the players' expected utilities at equilibrium are

$$V_e(\beta^*, \alpha^*) = \frac{(v_1 + D)(v_2 + D)}{v_1 + v_2 + 2D}; \quad V_o(\beta^*, \alpha^*) = -\frac{(w_1 + F)(w_2 + F)}{w_1 + w_2 + 2F}$$

if $0 < D < D_{crit}$, and

$$V_e(\beta^*, \alpha^*) = D; \quad V_o(\beta^*, \alpha^*) = -F$$

if $D_{crit} < D$ reflecting that, at a legal equilibrium, expected utilities are independent of site value parameters. For later reference, note that $D = D_{crit}$ is the minimum value of D satisfying

$$\frac{v_1}{v_1 + D} + \frac{v_2}{v_2 + D} \leq 1. \quad (3)$$

Now we return to the general (n, k) game and apply the same transformation—adding $(P, -F)$ to each outcome—to reduce the number of parameters. The utilities in the transformed game become $V_e(S, T)$ and $V_o(S, T)$, as follows:

$$V_e(S, T) = \begin{cases} D + \sum_{j \in S} v_j, & \text{if } S \cap T = \emptyset \\ 0, & \text{if } S \cap T \neq \emptyset. \end{cases} \quad (4)$$

$$V_o(S, T) = \begin{cases} -F - \sum_{j \in S} w_j, & \text{if } S \cap T = \emptyset \\ 0, & \text{if } S \cap T \neq \emptyset. \end{cases} \quad (5)$$

Recall the convention that a summation over an empty set equals 0.

When all parameter values are known, any (n, k) game can be represented in bimatrix form, and then Nash equilibria can be determined using software such as Gambit. However, the bimatrices grow much faster than n and k , so that it is difficult to assess what parameter values correspond to adequate penalties and compliance even when n and k are small. To gain insight, we turn to less general models.

3 The (n, k, m) game: Special cases

In this paper, we study a class of (n, k) games called (n, k, m) games. Let n , k , and m be positive integers with $k < n$ and $m \leq n$. An (n, k, m) game is an (n, k) game in which the n sites can be partitioned into m levels, such that every site at level r ($r = 1, \dots, m$) has Inspectee violation value v_r and Inspector violation cost $-w_r$. Lower-numbered levels are most important, so that $0 < v_m < v_{m-1} < \dots < v_2 < v_1$ and $0 < w_m < w_{m-1} < \dots < w_2 < w_1$. In fact, the defining feature of (n, k, m) games is that Inspector and Inspectee agree on the relative importance of any two sites. Let n_r denote the number of sites at level r ; of course, $n_r \geq 0$ for $r = 1, 2, \dots, m$, and $\sum_{r=1}^m n_r = n$.

In an (n, k, m) game, strategies can be simplified. A pure strategy for a player in any (n, k) game is a subset of sites. In such a strategy, one site could be replaced by another site with the same site values, or by a site randomly chosen from a subset of sites with identical site values, and both players' utilities would be unchanged. It follows that, in an (n, k, m) game, every pure strategy produces exactly the same utilities as a strategy that specifies the number of sites to be selected at random at each level.

Thus a pure strategy for Inspector in an (n, k, m) game is $\vec{j} = \langle j_1, j_2, \dots, j_m \rangle$, where j_1, j_2, \dots, j_m are integers and $0 \leq j_r \leq n_r$ for each r . The interpretation of \vec{j} is that, for each $r = 1, \dots, m$, j_r level r sites are to be chosen at random for inspection. (Throughout, we will use vector notation, such as \vec{j} , exclusively for m -vectors of which all components are non-negative integers.) As discussed above, we will assume that an Inspector strategy \vec{j} satisfies $\sum_{r=1}^m j_r = k$, because we wish to consider only strategies that are not weakly dominated. Below it will be convenient to think of any strategy $\vec{j} = \langle j_1, j_2, \dots, j_m \rangle$ as a pure strategy, but we note that in the context of the (n, k) game the Inspector strategy \vec{j} is pure iff $j_r = 0$ or $j_r = n_r$ for every r .

Similarly, we denote a pure strategy for Inspectee by $\vec{i} = \langle i_1, i_2, \dots, i_m \rangle$ where, for each r , i_r is an integer and $0 \leq i_r \leq n_r$. Again, we assume that $\sum_{r=1}^m i_r \leq n - k$ to ensure that the Inspectee strategy \vec{i} is not weakly dominated.

Consider the extreme case $m = 1$, in which all sites have equal value, say v for Inspectee and w for Inspector. Inspector's only (undominated) strategy is $\langle k \rangle$, i.e. inspect k sites chosen at random. Thus the $(n, k, 1)$ game reduces to a one-person decision problem in which Inspectee decides whether to violate at $0, 1, 2, \dots$, or $n - k$ sites.

The $(n, k, 1)$ game is therefore a non-zerosum version of the Point Catcher Game studied by Ruckle [1983]. It is easy to verify that, by violating at i sites, Inspectee achieves expected utility

$$\frac{\binom{n-i}{k}}{\binom{n}{k}} [D + iv],$$

where $i = 0, 1, \dots, n - k$. Moreover, if i_0 is the smallest value of i such that

$$\frac{D}{v} > \frac{n - k - 2i}{k}, \quad (6)$$

then the unique choice that maximizes Inspectee's expected utility is to violate at i_0 (randomly chosen) sites. (If $i = i_0$ produces equality in (6), then Inspectee is indifferent between violating at i_0 and $i_0 + 1$ sites.) In particular, Inspectee is completely deterred from violation (i.e. the only equilibria are legal) iff $D > \left(\frac{n-k}{k}\right)v$. Equivalently, if all n sites have Inspectee violation value v (and common Inspector violation cost), then all equilibria are legal iff

$$\frac{nv}{v + D} < k. \quad (7)$$

One justification for studying games of the (n, k, m) class is that, when m is small, these games model an approach to inspection that seems widely used in practice, at least as a first approximation: Group the sites into m classes according to strategic importance. For example, the Organization for the Prohibition of Chemical Weapons (OPCW) is reported to classify sites into $m = 3$ levels.

We now study an $(n, 1, 2)$ game in detail. Because $m = 2$, every site is *high-value*, with Inspectee violation value v_1 and Inspector violation cost w_1 , or *low-value*, with site value parameters v_2 and w_2 , where $0 < v_2 < v_1$ and $0 < w_2 < w_1$. There are $n_1 > 0$ high-value sites and $n_2 > 0$ low-value sites, and $n_1 + n_2 = n$. A strategy for Inspectee is $\langle i_1, i_2 \rangle$: Violate at i_1 randomly-chosen high-value sites and i_2 randomly-chosen low-value sites. As usual, $0 \leq i_1 \leq n_1$, $0 \leq i_2 \leq n_2$, and $i_1 + i_2 \leq n - 1$. Because $k = 1$, Inspector's two strategies are $\langle 1, 0 \rangle$ (inspect a randomly-chosen high-value site) and $\langle 0, 1 \rangle$ (inspect a randomly-chosen low-value site). A mixed strategy for Inspector is $(\alpha, 1 - \alpha) = \alpha \langle 1, 0 \rangle + (1 - \alpha) \langle 0, 1 \rangle$, where $0 \leq \alpha \leq 1$.

We now solve a specific $(5, 1, 2)$ game with $n_1 = 3$ and $n_2 = 2$. Since Inspectee has 11 pure strategies, the strategic form of this game is a 2×11 bimatrix, shown in Table 3. The entries of Table 3 were obtained using (4) and (5).

	$\langle 1, 0 \rangle$	$\langle 0, 1 \rangle$
$\langle 0, 0 \rangle$	$D, -F$	$D, -F$
$\langle 1, 0 \rangle$	$\frac{2}{3}(v_1 + D), -\frac{2}{3}(w_1 + F)$	$v_1 + D, -(w_1 + F)$
$\langle 2, 0 \rangle$	$\frac{1}{3}(2v_1 + D), -\frac{1}{3}(2w_1 + F)$	$2v_1 + D, -(2w_1 + F)$
$\langle 3, 0 \rangle$	$0, 0$	$3v_1 + D, -(3w_1 + F)$
$\langle 0, 1 \rangle$	$v_2 + D, -(w_2 + F)$	$\frac{1}{2}(v_2 + D), -\frac{1}{2}(w_2 + F)$
$\langle 1, 1 \rangle$	$\frac{2}{3}(v_1 + v_2 + D), -\frac{2}{3}(w_1 + w_2 + F)$	$\frac{1}{2}(v_1 + v_2 + D), -\frac{1}{2}(w_1 + w_2 + F)$
$\langle 2, 1 \rangle$	$\frac{1}{3}(2v_1 + v_2 + D), -\frac{1}{3}(2w_1 + w_2 + F)$	$\frac{1}{2}(2v_1 + v_2 + D), -\frac{1}{2}(2w_1 + w_2 + F)$
$\langle 3, 1 \rangle$	$0, 0$	$\frac{1}{2}(3v_1 + v_2 + D), -\frac{1}{2}(3w_1 + w_2 + F)$
$\langle 0, 2 \rangle$	$2v_2 + D, -(2w_2 + F)$	$0, 0$
$\langle 1, 2 \rangle$	$\frac{2}{3}(v_1 + 2v_2 + D), -\frac{2}{3}(w_1 + 2w_2 + F)$	$0, 0$
$\langle 2, 2 \rangle$	$\frac{1}{3}(2v_1 + 2v_2 + D), -\frac{1}{3}(2w_1 + 2w_2 + F)$	$0, 0$

Table 3. A $(5, 1, 2)$ game with $n_1 = 3$ high-value sites and $n_2 = 2$ low-value sites

As discussed earlier, strategies that are not in Table 3 are known to be dominated. But there may be other dominated strategies, such as Inspectee's strategy $\langle 2, 2 \rangle$.

Our principal objective is to determine the values of D and the Inspector strategies that ensure a legal equilibrium. Thus, we ask when and how can it be guaranteed that Inspectee's only rational choice is to play the pure strategy $\langle 0, 0 \rangle$?

To create a specific example, assume that $v_1 = 4, v_2 = 2, w_1 = 2, w_2 = 1$, and $F = 1$. Table 4 shows the resulting game matrix, which depends on the remaining parameter, D .

	$\langle 1, 0 \rangle$	$\langle 0, 1 \rangle$
$\langle 0, 0 \rangle$	$D, -1$	$D, -1$
$\langle 1, 0 \rangle$	$(2/3)(4 + D), -2$	$4 + D, -3$
$\langle 2, 0 \rangle$	$(1/3)(8 + D), -5/3$	$8 + D, -5$
$\langle 3, 0 \rangle$	$0, 0$	$12 + D, -7$
$\langle 0, 1 \rangle$	$2 + D, -2$	$(1/2)(2 + D), -1$
$\langle 1, 1 \rangle$	$(2/3)(6 + D), -8/3$	$(1/2)(6 + D), -2$
$\langle 2, 1 \rangle$	$(1/3)(10 + D), -2$	$(1/2)(10 + D), -3$
$\langle 3, 1 \rangle$	$0, 0$	$(1/2)(14 + D), -4$
$\langle 0, 2 \rangle$	$4 + D, -3$	$0, 0$
$\langle 1, 2 \rangle$	$(2/3)(8 + D), -10/3$	$0, 0$
$\langle 2, 2 \rangle$	$(1/3)(12 + D), -7/3$	$0, 0$

Table 4. A specific $(5, 1, 2)$ game $n_1 = 3$ and $n_2 = 2$.

The game of Table 4 is a non-zerosum (in fact, non-constantsum) game, but a technique usually applied only to zerosum games can answer our questions. First, it is straightforward to verify that there is never a “pure” equilibrium [in the sense of (n, k, m) games]. Therefore an equilibrium strategy for Inspector must be mixed, i.e. $(\alpha, 1 - \alpha)$ where $0 < \alpha < 1$. We noted above that, at equilibrium, Inspectee chooses either certain compliance, $\langle 0, 0 \rangle$, or a non-trivial mixed strategy. It is a standard result that this mixture can be taken to be a probability distribution $(\beta, 1 - \beta)$ over some pair of rows, $\langle i_1^1, i_2^1 \rangle$ and $\langle i_1^2, i_2^2 \rangle$, with equilibrium values of β determined by Inspector’s utilities. But illegal equilibria are not our main focus. The values of D and α that correspond to a legal equilibrium can be determined without reference to Inspector’s utilities.

Figure 1 shows the expected utility to Inspectee of each of the 11 pure strategies in Table 3, as a function of α , when $D = 15$. The upper envelope of these functions is highlighted.

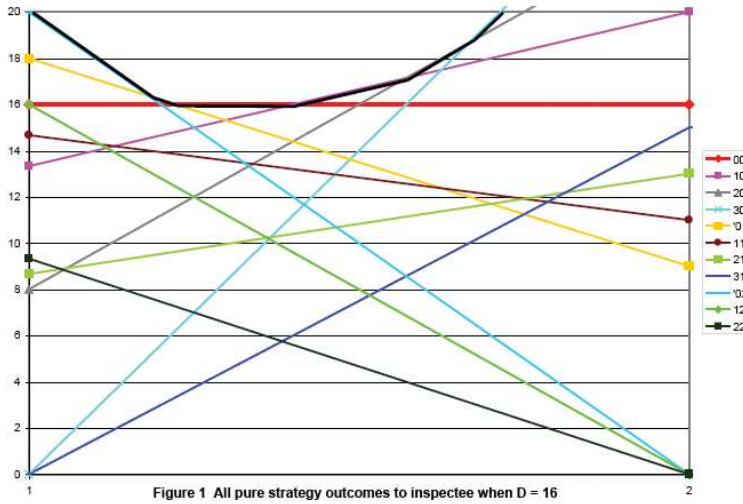


Figure 1. Inspectee's Expected Utilities when $D = 15$ in the $(5, 1, 2)$ example.

For any value of α , Inspectee maximizes its expected utility by selecting a strategy that coincides with the upper envelope in Figure 1. Thus any Nash Equilibrium must lie on this envelope. Inspector prefers outcomes where Inspectee complies to all other outcomes, so at equilibrium it selects a value of α corresponding to a point on the upper envelope associated with Inspectee strategy $(0, 0)$, if any such point is available. It follows that, in the case shown in Figure 1, the only equilibria of the game must correspond to values of α associated with the horizontal segment of the upper envelope. Moreover, any such α determines a legal equilibrium.

Figure 2 presents the same set of expected utilities as Figure 1, but when $D = 8$. (The vertical scale has been expanded slightly.)

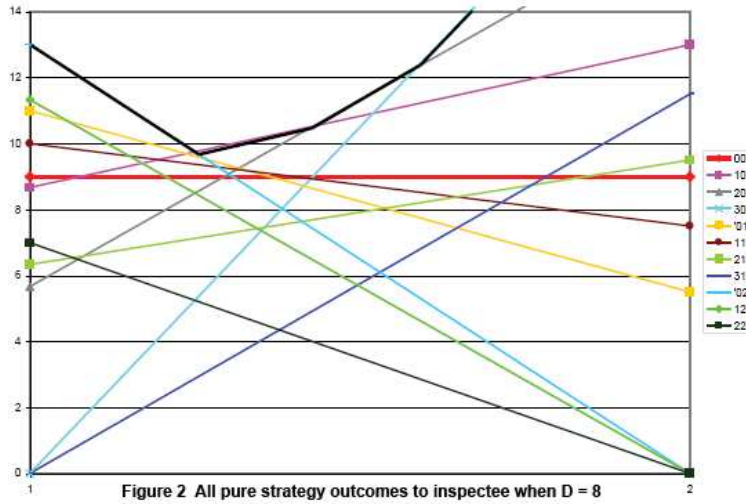


Figure 2. Inspectee's Expected Utilities when $D = 8$ in the $(5, 1, 2)$ example.

In this case, the (horizontal) line corresponding to Inspectee strategy $\langle 0, 0 \rangle$ is never part of the upper envelope, so no legal equilibrium is available. At equilibrium, Inspector must choose the specific value of α that corresponds to the minimum point on the upper envelope. This value is unique, since the upper envelope has no horizontal segments.

As D increases, the horizontal line corresponding to Inspectee strategy $\langle 0, 0 \rangle$ moves upward. Thus, between $D = 15$ and $D = 8$ there is a critical value of D , which we call D_{crit} , such that if $D > D_{crit}$, all Nash equilibria are legal, while if $D < D_{crit}$, all Nash equilibria are illegal. Figure 3 shows the same diagram as above for $D = D_{crit} = 12.55$.

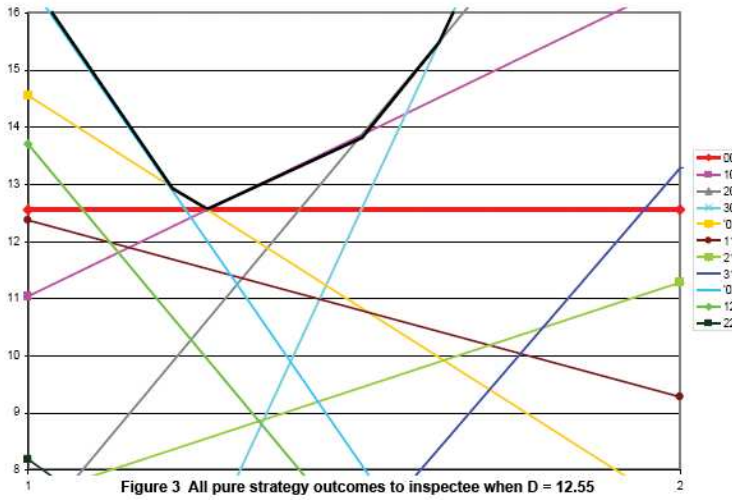


Figure 3. Inspectee's Expected Utilities when $D = D_{crit} = 12.55$ in the (5, 1, 2) example.

In the next section, we derive expressions for D_{crit} and α as functions of the parameters of the game. For completeness, we show in Table 5 Inspectee and Inspector strategies at all generic Nash equilibria depend on D in Table 5. Inspectee strategies (values of β) were computed by equating Inspector utilities.

D	Inspectee strategy	Inspector strategy
$D \geq 12.5498$	$\langle 0, 0 \rangle$	$\frac{12}{4+D} \leq \alpha \leq \frac{D-2}{D+2}$
$12.5498 \geq D \geq 10$	$0.471\langle 0, 1 \rangle + 0.529\langle 1, 0 \rangle$	$\alpha = \frac{18+3D}{14+5D}$
$10 \geq D \geq 6$	$0.75\langle 1, 0 \rangle + 0.25\langle 0, 2 \rangle$	$\alpha = \frac{3}{4}$
$6 \geq D \geq 4.745$	$0.384\langle 1, 0 \rangle + 0.616\langle 1, 1 \rangle$	$\alpha = \frac{6+3D}{14+3D}$
$4.745 \geq D \geq 2$	$0.828\langle 1, 1 \rangle + 0.172\langle 2, 0 \rangle$	$\alpha = \frac{30+3D}{38+5D}$
$2 \geq D > 0$	$0.5\langle 2, 0 \rangle + 0.5\langle 1, 2 \rangle$	$\alpha = \frac{3}{4}$

Table 5. Equilibrium strategies in the (5, 1, 2) example.

The evolution of Inspectee's optimal strategy as D decreases is interesting. Inspectee is deterred from violating whenever D exceeds $D_{crit} \approx 12.55$. But once D falls below this critical value, Inspectee violates for certain. Initially, violations occur at just one site, which may be high-value or low-value. As the effective penalty for detected violation, D , decreases, Inspectee takes greater risks by violating at more sites. The next step is to mix one high-value site with both low-value sites, then to switch to a strategy of always violating at a high-value site and sometimes also violating at a low-value site, then to violate at two sites—either one at each level or two of high value. When penalties are very low, the loss of

benefit from not violating is so significant that Inspectee always violates at one high-value site, and adds either another high-value site or both low-value sites.

Figure 4 shows the equilibrium values of α as a function of D for the $(5, 1, 2)$ example game. The phenomenon called the “cone of deterrence” [Kilgour, 1992] is clearly visible on the right.

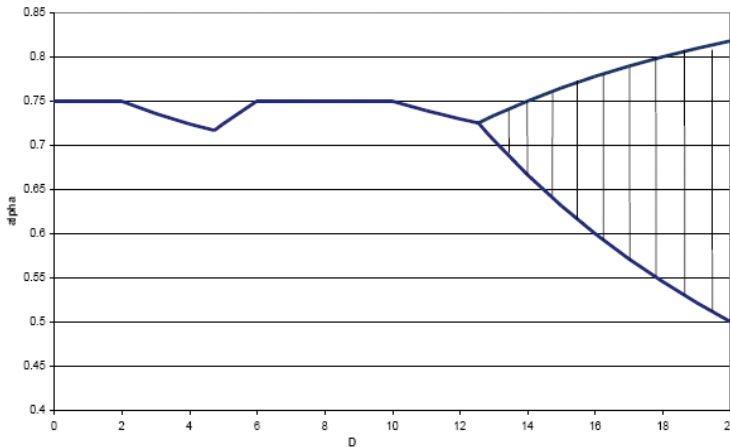


Figure 4 Equilibrium values of alpha as a function of D

Figure 4. Equilibrium values of α as a function of D in the $(5, 1, 2)$ example.

4 Analysis

4.1 The $(n, 1, m)$ Game

In this section, we find the legal equilibria for (n, k, m) games with $k = 1$ inspection, such as the example in the previous section. To avoid trivialities, we assume henceforth that $m > 1$ and that $n_r \geq 1$ for $r = 1, 2, \dots, m$, i.e. there are at least two levels, with at least one site at each level.

In an $(n, 1, m)$ game, Inspector’s m (undominated) pure strategies correspond to the level of the site to be inspected. Therefore a mixed Inspector strategy can be denoted $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, where α_r is the probability that the inspected site is at level r . Recall that a pure Inspectee strategy is $\vec{i} = \langle i_1, i_2, \dots, i_m \rangle$, where i_r is an integer satisfying $0 \leq i_r \leq n_r$ for each r , and $\sum_{r=1}^m i_r \leq n - 1$. (Recall that we use vector notation only for m -vectors with non-negative integer components.) Define the m -vectors $\vec{0} = \langle 0, 0, \dots, 0 \rangle$ and

$\vec{e}_r = \langle 0, 0, \dots, 0, 1, 0, \dots, 0 \rangle$ (where the r^{th} component equals 1). Thus, \vec{e}_r is the Inspectee strategy of violating at exactly one level r site and nowhere else. Note that $\vec{i} = \sum_{r=1}^m i_r \vec{e}_r$.

It follows from (4) that Inspectee's expected utility can be written

$$\begin{aligned} H(\vec{i}, \alpha) &= \left(\sum_{r=1}^m \left(1 - \frac{i_r}{n_r} \right) \alpha_r \right) \left(D + \sum_{r=1}^m i_r v_r \right) \\ &= \left(1 - \sum_{r=1}^m \frac{i_r}{n_r} \alpha_r \right) \left(D + \sum_{r=1}^m i_r v_r \right). \end{aligned} \quad (8)$$

By (8), $\vec{i} = \vec{0}$ implies $H(\vec{0}, \alpha) = D$ for any α . Clearly, a necessary condition for $\langle \vec{0}, \alpha \rangle$ to be a legal equilibrium is $H(\vec{e}_r, \alpha) \leq D$ for every r . Because $H(\vec{e}_r, \alpha) = \left(1 - \frac{\alpha_r}{n_r} \right) (D + v_r)$, it is easy to verify that this inequality is equivalent to $\alpha_r \geq \frac{n_r v_r}{v_r + D}$. This observation, and the analogy with (3) and (7), motivates the next definition.

Definition 1. D_{crit} is the unique positive value of D that solves

$$\sum_{r=1}^m \frac{n_r v_r}{v_r + D} = 1.$$

Note that D_{crit} is well-defined, because the left side of this equation is a monotonically decreasing function of D that equals n at $D = 0$ and approaches 0 as $D \rightarrow \infty$.

Theorem 2. If $D < D_{crit}$ in an $(n, 1, m)$ game, all equilibria are illegal.

Proof: Assume $D < D_{crit}$ and fix any mixed Inspector strategy, α . We show that $\alpha_r < \frac{n_r v_r}{v_r + D}$ for at least one value of r , which will prove the theorem because it implies that $\left(1 - \frac{\alpha_r}{n_r} \right) (D + v_r) > D$ which, by (8), is equivalent to $H(\vec{e}_r, \alpha) > H(\vec{0}, \alpha)$. Thus, Inspectee's expected utility for violating at exactly one level r site exceeds its utility for compliance, so compliance cannot be an equilibrium strategy for Inspectee.

Without loss of generality, assume that $\alpha_r \geq \frac{n_r v_r}{v_r + D}$ for $r = 1, 2, \dots, m-1$. Since $D < D_{crit}$, it follows from Definition 1 that $\sum_{r=1}^m \frac{n_r v_r}{v_r + D} > 1 = \sum_{r=1}^m \alpha_r$, which is equivalent to $\sum_{r=1}^{m-1} \left(\frac{n_r v_r}{v_r + D} - \alpha_r \right) > \alpha_m - \frac{n_m v_m}{v_m + D}$. But by assumption each term in the summation is at most zero, so that $0 > \alpha_m - \frac{n_m v_m}{v_m + D}$. Therefore $\frac{n_m v_m}{v_m + D} > \alpha_m$. ■

Definition 2. A bounded Inspector strategy is a mixed Inspector strategy, α , satisfying $\alpha_r \geq \frac{n_r v_r}{v_r + D}$ for each $r = 1, 2, \dots, m$.

It was observed in the proof of Theorem 2 that bounded Inspector strategies cannot exist when $D < D_{crit}$. But they must exist when $D \geq D_{crit}$, because $\sum_{r=1}^m \frac{n_r v_r}{v_r + D} \leq 1$ by

Definition 1, so α can be chosen so that all conditions of Definition 2 are met. As the next theorem shows, the existence of a bounded Inspector strategy guarantees the existence of a legal equilibrium.

Theorem 3. *If $D > D_{crit}$ in an $(n, 1, m)$ game, all equilibria are of the form $(\vec{0}, \alpha)$ where α is a bounded Inspector strategy. Conversely, any bounded Inspector strategy induces a legal equilibrium.*

Proof: First observe that, if Inspectee complies, Inspector's expected utility is always $-F$ by (5), so any strategy maximizes Inspector's expected utility. To complete the proof we must show that, whenever $D > D_{crit}$, compliance maximizes Inspectee's expected utility iff Inspector chooses a bounded Inspector strategy. To this end, assume $D > D_{crit}$ and fix a bounded Inspector strategy, α .

Now select any integers r and i_r such that $1 \leq r \leq m$ and $1 \leq i_r \leq n_r$. We show that $i_r \vec{e}_r$ cannot be Inspectee's unique best response to α . By (8),

$$H(i_r \vec{e}_r, \alpha) = \left(1 - \frac{i_r \alpha_r}{n_r}\right) (i_r v_r + D) \leq \left(1 - \frac{i_r v_r}{v_r + D}\right) (i_r v_r + D),$$

because $\alpha_r \geq \frac{n_r v_r}{v_r + D}$. Therefore,

$$H(i_r \vec{e}_r, \alpha) \leq D + i_r v_r \left(1 - \frac{D}{v_r + D} - \frac{i_r v_r}{v_r + D}\right).$$

But $1 - \frac{D}{v_r + D} - \frac{i_r v_r}{v_r + D} \leq 1 - \frac{D}{v_r + D} - \frac{v_r}{v_r + D} = 0$, which shows that $H(i_r \vec{e}_r, \alpha) \leq 0 = H(\vec{0}, \alpha)$. Thus, facing a bounded Inspector strategy, no strategy of the form $i_r \vec{e}_r$ with $i_r \geq 1$ is better for Inspectee than compliance.

Therefore if Inspectee has a best response $\vec{i} \neq \vec{0}$ to α , there must exist two integers, r and s , such that $1 \leq r < s \leq m$ and both $i_r > 0$ and $i_s > 0$. We show that no such strategy can be better than compliance by showing that Inspectee's expected utility is a

strictly decreasing function of i_s . To this end, we differentiate (8) formally to obtain

$$\begin{aligned}
\frac{\partial H(\vec{i}, \alpha)}{\partial i_s} &= -\frac{\alpha_s}{n_s} \left(D + \sum_{t=1}^m i_t v_t \right) + \left(1 - \sum_{t=1}^m \frac{i_t}{n_t} \alpha_t \right) v_s \\
&\leq -\frac{v_s}{v_s + D} \left(D + \sum_{t=1}^m i_t v_t \right) + \left(1 - \sum_{t=1}^m \frac{i_t v_t}{v_t + D} \right) v_s \\
&= v_s \left(\frac{v_s}{v_s + D} - \sum_{t=1}^m \frac{i_t v_t}{v_t + D} - \sum_{t=1}^m \frac{i_t v_t}{v_s + D} \right) \\
&\leq v_s \left(\frac{v_s}{v_s + D} - \sum_{t=1}^{s-1} \frac{i_t v_t}{v_t + D} - \sum_{t=s}^m \frac{i_t v_t}{v_s + D} - \sum_{t=1}^m \frac{i_t v_t}{v_s + D} \right) \\
&= \frac{v_s}{v_s + D} \left(v_s - \sum_{t=1}^{s-1} \left[\frac{v_s + D}{v_t + D} + 1 \right] i_t v_t - 2 \sum_{t=s}^m i_t v_t \right).
\end{aligned}$$

But now $i_r \geq 1$ implies that

$$\frac{\partial H(\vec{i}, \alpha)}{\partial i_s} \leq \frac{v_s}{v_s + D} \left(v_s - \left[\frac{v_s + D}{v_r + D} + 1 \right] i_r v_r \right) \leq \frac{v_s}{v_s + D} (v_s - v_r) < 0,$$

since $\frac{v_s + D}{v_r + D} > 0$ and $v_s < v_r$. Thus \vec{i} cannot maximize $H(\vec{i}, \alpha)$ if $i_s > 0$, completing the proof. ■

In the previous section, we identified Inspector's strategies at all mixed-strategy equilibria for all $D \geq 0$ in a particular $(5, 1, 2)$ game. Motivated by graphics such as Figures 1 - 3, we reduced D iteratively and observed where Inspector's utility for compliance first met the minimal point on the upper envelope. Three lines meet at such a transition point, a fact that determines the threshold value of D . Definition 1, and our algebraic results based on it, is the generalization to the single inspection ($k = 1$) case. Bounded Inspector strategies α generalize the "cone of deterrence" illustrated in Figure 4.

4.2 The (n, k, m) Game when $k > 1$: General Considerations

We now investigate the legal equilibria of the (n, k, m) inspection game when $k > 1$. Note that (n, k) games have no time dimension, so the fact that there are $k > 1$ inspections means that k different sites will be inspected simultaneously; detection occurs iff there is a violation at any inspected site. Intuitively, we expect that as the number of inspections increases, so does Inspector's capacity to deter violations.

Recall that a typical pure Inspector strategy is denoted $\vec{j} = \langle j_1, j_2, \dots, j_m \rangle$. We now assume that the number of inspections is small relative to the number of sites at each level,

i.e. that $k \leq n_r$ for all r . Then any sequence of non-negative integers j_1, j_2, \dots, j_m satisfying $\sum_{r=1}^m j_r = k$ defines a pure Inspector strategy. There are $\mu = \binom{k+m-1}{k}$ such strategies, which we enumerate $\vec{j}^1, \dots, \vec{j}^\mu$. Writing each strategy as a column vector, we form an m by μ array, so that a mixed Inspector strategy can be denoted $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\mu)$, where α_ℓ is the probability that pure strategy \vec{j}^ℓ , the ℓ^{th} column of the array, is selected.

If Inspectee plays pure strategy $\vec{i} = \langle i_1, i_2, \dots, i_m \rangle$ and Inspector plays pure strategy \vec{j} , then the probability that no violation is detected equals

$$\delta(\vec{i}, \vec{j}) = \frac{\binom{n_1-i_1}{j_1}}{\binom{n_1}{j_1}} \frac{\binom{n_2-i_2}{j_2}}{\binom{n_2}{j_2}} \dots \frac{\binom{n_m-i_m}{j_m}}{\binom{n_m}{j_m}}, \quad (9)$$

where $\binom{n-i}{j} = 0$ whenever $j > n - i$. If Inspectee complies, $\vec{i} = \vec{0}$, and clearly $\delta(\vec{0}, \vec{j}) = 1$. It follows from (4) and (9) that, if Inspector plays mixed strategy α , Inspectee's expected utility for playing \vec{i} is

$$H(\vec{i}, \alpha) = \left(\sum_{\ell=1}^{\mu} \alpha_\ell \delta(\vec{i}, \vec{j}^\ell) \right) \left(D + \sum_{r=1}^m i_r v_r \right). \quad (10)$$

We now prove a useful lemma.

Lemma 1. For $r = 1, 2, \dots, m$, $H(\vec{e}_r, \alpha) \leq H(\vec{0}, \alpha)$ iff $\sum_{\ell=1}^{\mu} \alpha_\ell j_r^\ell \geq \frac{n_r v_r}{v_r + D}$, with equality iff $\sum_{\ell=1}^{\mu} \alpha_\ell j_r^\ell = \frac{n_r v_r}{v_r + D}$.

Proof: For any $r = 1, 2, \dots, m$, it follows from (10) that

$$\begin{aligned} H(\vec{e}_r, \alpha) &= (v_r + D) \sum_{\ell=1}^{\mu} \alpha_\ell \delta(\vec{e}_r, \vec{j}^\ell) = (v_r + D) \sum_{\ell=1}^{\mu} \alpha_\ell \left(1 - \frac{j_r^\ell}{n_r} \right) \\ &= (v_r + D) \left(1 - \frac{1}{n_r} \sum_{\ell=1}^{\mu} \alpha_\ell j_r^\ell \right). \end{aligned}$$

From (10), it is also clear that $H(\vec{0}, \alpha) = D$; the lemma now follows easily. ■

Lemma 1 implies that the existence of an Inspector mixed strategy, α , with the property that $H(\vec{e}_r, \alpha) = H(\vec{0}, \alpha)$ for all values of r would imply that

$$\sum_{r=1}^m \frac{n_r v_r}{v_r + D} = \sum_{r=1}^m \sum_{\ell=1}^{\mu} j_r^\ell \alpha_\ell = \sum_{\ell=1}^{\mu} \alpha_\ell \left(\sum_{r=1}^m j_r^\ell \right) = k \sum_{\ell=1}^{\mu} \alpha_\ell = k. \quad (11)$$

This observation, and the analogy with (3) and (7), motivates us to generalize Definition 1.

Definition 3. D_{crit} is the positive value of D that solves (11).

Similar to Definition 1, Definition 3 specifies D_{crit} uniquely because the left side of (11) is a strictly decreasing function of D that equals $n > k$ when $D = 0$ and approaches 0 as $D \rightarrow \infty$.

We can now show that $D \geq D_{crit}$ is a necessary condition for a legal equilibrium.

Theorem 4. *If $D < D_{crit}$ in an (n, k, m) game, all equilibria are illegal.*

Proof: Assume that $D < D_{crit}$ and fix a mixed Inspector strategy, α . We show that $\sum_{\ell=1}^{\mu} \alpha_{\ell} j_r^{\ell} < \frac{n_r v_r}{v_r + D}$ for at least one value of r . Then Lemma 1 will prove that $H(\vec{e}_r, \alpha) > H(\vec{0}, \alpha)$, so that compliance cannot be an equilibrium strategy for Inspectee.

Assume without loss of generality that $\sum_{\ell=1}^{\mu} \alpha_{\ell} j_r^{\ell} \geq \frac{n_r v_r}{v_r + D}$ for $r = 1, 2, \dots, m-1$. Since $D < D_{crit}$, it follows from Definition 3 that $\sum_{r=1}^m \frac{n_r v_r}{v_r + D} > k$. Therefore

$$k = \sum_{\ell=1}^{\mu} \alpha_{\ell} \sum_{r=1}^m j_r^{\ell} = \sum_{r=1}^m \sum_{\ell=1}^{\mu} \alpha_{\ell} j_r^{\ell} \geq \sum_{r=1}^{m-1} \frac{n_r v_r}{v_r + D} + \sum_{\ell=1}^{\mu} \alpha_{\ell} j_m^{\ell}.$$

It follows that

$$\sum_{\ell=1}^{\mu} \alpha_{\ell} j_m^{\ell} \leq k - \sum_{r=1}^{m-1} \frac{n_r v_r}{v_r + D} < \sum_{r=1}^m \frac{n_r v_r}{v_r + D} - \sum_{r=1}^{m-1} \frac{n_r v_r}{v_r + D} = \frac{n_m v_m}{v_m + D},$$

completing the proof. ■

Thus a necessary condition for a legal equilibrium is that the expected number of inspections at each level, r , is not less than $\frac{n_r v_r}{v_r + D}$. We now show that, at least when m is small, this condition is also sufficient.

4.3 The (n, k, m) Game when $k > 1$ and $m = 2$ or $m = 3$

We now analyze the $(n, k, 2)$ and $(n, k, 3)$ games. These cases are of special interest because of the tendency, noted above, for inspectorates to classify sites into two or three categories, and then to treat all sites in each category as (approximately) identical. Our objective here is to prove that legal equilibria occur whenever $D > D_{crit}$ when $m = 2$ and when $m = 3$.

Recall that in Definition 3, D_{crit} is defined to be the value of D that solves

$$\sum_{r=1}^m \frac{n_r v_r}{v_r + D} = k.$$

Also, Inspector's pure strategies are $\vec{j}^1, \dots, \vec{j}^{\mu}$; a typical mixed Inspector strategy is $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\mu})$, where α_{ℓ} is the probability that pure strategy \vec{j}^{ℓ} is selected.

The pure Inspectee strategy, \vec{e}_r , specifies that violation is to take place at one randomly-chosen level r site and nowhere else. Clearly, a necessary condition for $(\vec{0}, \alpha)$ to be an equilibrium is $H(\vec{e}_r, \alpha) \leq H(\vec{0}, \alpha)$ which, by Lemma 1, is equivalent to $\sum_{\ell=1}^{\mu} \alpha_{\ell} j_r^{\ell} \geq \frac{n_r v_r}{v_r + D}$. We begin by constructing a mixed Inspector strategy α that satisfies all of these conditions for $r = 1, \dots, m$.

Suppose that the vector $u = (u_1, u_2, \dots, u_m) \in \mathbb{R}^m$ satisfies $\sum_{r=1}^m u_r = k$ and $u_r \geq 0$ for all r . A mixed Inspector strategy α is called a u -strategy iff $\sum_{\ell=1}^{\mu} \alpha_{\ell} j_r^{\ell} = u_r$ for each r and, of course, $\sum_{\ell=1}^{\mu} \alpha_{\ell} = 1$. First we show how to construct a particular u -strategy called $\tilde{\alpha}(u)$.

Fix a vector u satisfying the conditions above, and define $\vec{h}(u) = \vec{h}$ by $\vec{h} = \langle h_1, \dots, h_m \rangle$ where $h_r = \lfloor u_r \rfloor$ for $r = 1, 2, \dots, m$. Then the components of $u - \vec{h}$ satisfy $0 \leq u_r - h_r < 1$; denote their sum by γ and note that $\gamma = k - \sum_{r=1}^m h_r$. In fact, γ must be an integer, and must satisfy $0 \leq \gamma < m$. Let $B = \binom{m}{\gamma}$, and enumerate $\vec{\omega}_1, \vec{\omega}_2, \dots, \vec{\omega}_B$ the set of all binary m -vectors whose components sum to γ .

Lemma 2. *There exist non-negative numbers $\kappa_1, \kappa_2, \dots, \kappa_B$ satisfying $u - \vec{h} = \sum_{b=1}^B \kappa_b \vec{\omega}^b$ and $\sum_{b=1}^B \kappa_b = 1$.*

Proof: See Zhao et al., [2007]. ■

Now define $\vec{\phi}^b = \vec{h} + \vec{\omega}^b$ for $b = 1, 2, \dots, B$, and let ϕ_{br} denote the r^{th} component of $\vec{\phi}^b$. Each $\vec{\phi}^b$ is a pure Inspector strategy, as the sum of its entries is $\sum_{r=1}^m \phi_{br} = \sum_{r=1}^m (h_r + \omega_{br}) = \sum_{r=1}^m h_r + \gamma = k$. Let $\Phi = \{\vec{\phi}^1, \vec{\phi}^2, \dots, \vec{\phi}^B\}$ and define the mixed Inspector strategy $\tilde{\alpha}(u) = \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_{\mu})$ as follows: for each $\ell = 1, \dots, \mu$, if $j^{\ell} \notin \Phi$, $\tilde{\alpha}_{\ell} = 0$; otherwise, $j^{\ell} = \vec{\phi}^b$ for some $b = 1, 2, \dots, B$, and $\tilde{\alpha}_{\ell} = \kappa_b$. It follows that $\tilde{\alpha}_{\ell} \geq 0$ for all ℓ , $\sum_{\ell=1}^{\mu} \tilde{\alpha}_{\ell} = \sum_{b=1}^B \kappa_b = 1$ and, for $r = 1, \dots, m$, $\sum_{\ell=1}^{\mu} \tilde{\alpha}_{\ell} j_r^{\ell} = \sum_{b=1}^B \kappa_b \phi_{br} = \sum_{b=1}^B \kappa_b (h_r + \omega_{br}) = h_r + \sum_{b=1}^B \kappa_b \omega_{br} = u_r$ using Lemma 2. Therefore, $\tilde{\alpha}$ is a u -strategy.

A vector $u = (u_1, u_2, \dots, u_m) \in \mathbb{R}^m$ is called *sufficient* iff it satisfies $\sum_{r=1}^m u_r = k$ and $u_r \geq \frac{n_r v_r}{v_r + D}$ for all r .

Lemma 3. *A sufficient vector u exists iff $D \geq D_{crit}$.*

Proof: Follows directly from Definition 3. ■

Theorem 4 and Definition 3 imply that, if $(\vec{0}, \alpha)$ is a legal equilibrium, then α is a u -strategy for a sufficient vector, u . Moreover, u can be determined by the relation $u_r = \sum_{\ell=1}^{\mu} \alpha_{\ell} j_r^{\ell}$, which must hold for each r . Our objective now is to prove a converse: If u is sufficient, then $(\vec{0}, \tilde{\alpha}(u))$ is an equilibrium.

We first establish a technical result that will be used later.

Lemma 4. Suppose $x, y, z, a, b,$ and c are real numbers that satisfy $0 \leq x, y, z \leq 1, x+y+z = 1,$ and $0 < a, b, c < 1.$ Then

$$\frac{bx + ay}{bcx + acy + abz} \geq x + y.$$

Proof: Observe that

$$\frac{bx + ay}{bcx + acy + abz} - x - y > \frac{bx + ay}{bx + ay + abz} - x - y = \frac{bx + ay - (x + y)(bx + ay + abz)}{bx + ay + abz}.$$

The denominator of the right-hand fraction is clearly positive. To show that the fraction itself is positive, we note that the numerator equals

$$\begin{aligned} bx + ay - (x + y)(bx + ay + abz) &= bx + ay - bx^2 - axy - abzx - bxy - ay^2 - abzy \\ &= bx(1 - x - y) + ay(1 - x - y) - abzx - abzy \\ &= bxz + ayz - abzx - abzy = z(bx + ay - abx - aby) \\ &= z(bx(1 - a) + ay(1 - b)) \geq 0, \end{aligned}$$

completing the proof. ■

To study the effects of a mixed inspector strategy of the form $\tilde{\alpha}(u)$, we need a new representation of $\delta \left(\vec{i}, \vec{j} \right)$ as defined by (9). First, let n and i be integers such that $0 < i < n.$ Define $S_0^n(i) = 0.$ For $j = 1, \dots, n - i,$ define

$$\begin{aligned} S_j^n(i) &= \frac{\binom{n-i}{j}}{\binom{n}{j}} = \frac{(n-i)(n-1-i) \dots (n-j+1-i)}{n(n-1) \dots (n-j+1)} \\ &= \left(1 - \frac{i}{n}\right) \left(1 - \frac{i}{n-1}\right) \dots \left(1 - \frac{i}{n-j+1}\right). \end{aligned}$$

Now set $s_\lambda^n(i) = 1 - \frac{i}{n-\lambda+1}$ for $\lambda = 1, 2, \dots, n - i.$ Then $S_j^n(i) = s_1^n(i)s_2^n(i) \dots s_j^n(i).$ Note that $S_j^n(i)$ can be thought of as a polynomial of degree j in $i,$ with zeroes at $n, n - 1, \dots,$ and $n - j + 1.$ As well, $S_j^n(0) = 1.$

Now fix integers n and j such that $0 \leq j < n,$ and temporarily think of i as a continuous variable. Then $S_j^n(i)$ is monotonically decreasing function of i for $0 \leq i \leq n - j + 1,$ as shown by the next result.

Lemma 5. For integers j and n such that $0 \leq j < n,$ $\frac{dS_j^n(i)}{di} \leq -\frac{j}{n}S_j^n(i)$ for $i \in [0, n - j + 1].$

Proof: The lemma is immediate when $j = 0,$ because $S_0^n(i) = 1.$ If $j > 0,$

$$\frac{dS_j^n(i)}{di} = -S_j^n(i) \left[\frac{1}{s_1^n(i)} \frac{1}{n} + \frac{1}{s_2^n(i)} \frac{1}{n-1} + \dots + \frac{1}{s_j^n(i)} \frac{1}{n-j+1} \right] \leq -\frac{j}{n}S_j^n(i)$$

since $\frac{ds_\lambda^n(i)}{di} = -\frac{1}{n-\lambda+1}$ and $s_\lambda^n(i) \cdot (n-\lambda+1) = n-\lambda+1-i \leq n$. ■

It follows from (9) that

$$\delta(\vec{i}, \vec{j}) = \prod_{r=1}^m S_{j_r}^{n_r}(i_r) = \prod_{r=1}^m \prod_{\lambda=1}^{j_r} s_\lambda^{n_r}(i_r). \quad (12)$$

In particular, $\delta(\vec{i}, \vec{j})$ can be thought of as a polynomial in i_1, i_2, \dots, i_m . In fact, it is a product of $\prod_{r=1}^m i_r$ monomials of the form $s_\lambda^n(i) = 1 - \frac{i}{n-\lambda+1}$. Of course, $\delta(\vec{0}, \vec{j}) = 1$.

For a fixed Inspectee strategy \vec{i} , let $P = P(\vec{i}) = D + \sum_{r=1}^m i_r v_r$. Then (10) and (12) imply that

$$H(\vec{i}, \alpha) = P \cdot \sum_{\ell=1}^{\mu} \alpha_\ell \prod_{r=1}^m S_{j_r^\ell}^{n_r}(i_r). \quad (13)$$

We now use (13) to analyze the consequences of the mixed Inspector strategy we constructed above, $\tilde{\alpha}$.

Lemma 6. *Suppose $D \geq D_{crit}$, u is sufficient, $1 \leq r \leq m \leq 3$ and the pure Inspectee strategy \vec{i} satisfies $i_t > 0$ for some $t = 1, 2, \dots, r$. Then $\frac{\partial H}{\partial i_r}(\vec{i}, \tilde{\alpha}(u)) \leq 0$.*

Proof: Substitute $\tilde{\alpha}(u) = \tilde{\alpha} = \sum_{b=1}^B \kappa_b \vec{\phi}^b$ into (13) to obtain

$$H(\vec{i}, \tilde{\alpha}) = P \cdot \sum_{b=1}^B \kappa_b \prod_{r=1}^m S_{\phi_{br}}^{n_r}(i_r). \quad (14)$$

For $b = 1, 2, \dots, B$ and $r = 1, 2, \dots, m$, define

$$q_{br} = \begin{cases} 1 & \text{if } \omega_{br} = 0 \\ s_1^{n_r}(i_r) & \text{if } \omega_{br} = 1. \end{cases}$$

Because $\phi_{br} = h_r + \omega_{br}$, we can extract a common factor of $C = \prod_{r=1}^m S_{h_r}^{n_r}(i_r)$ from the summation in (14) to obtain $H(\vec{i}, \tilde{\alpha}) = P \cdot C \cdot Q$ where

$$Q = \sum_{b=1}^B \kappa_b \prod_{r=1}^m q_{br}.$$

Note that P , C , and Q are all positive and hence their product is positive. Now think of $H(\vec{i}, \tilde{\alpha})$ as a polynomial in i_r and differentiate to obtain

$$\frac{\partial H(\vec{i}, \tilde{\alpha})}{\partial i_r} = CQv_r + PQ \frac{C}{S_{h_r}^{n_r}} \frac{dS_{h_r}^{n_r}}{di_r} + PC \frac{\partial Q}{\partial i_r} = CQP \left[\frac{v_r}{P} + \frac{1}{S_{h_r}^{n_r}} \frac{dS_{h_r}^{n_r}}{di_r} + \frac{1}{Q} \frac{\partial Q}{\partial i_r} \right]$$

Denote the expression in square brackets by R , and observe that, to prove the Lemma, we need only show that $R < 0$. By Lemma 5, $\frac{1}{S^{n_r}} \frac{dS^{n_r}}{di_r} \leq -\frac{h_r}{n_r}$. Also $\frac{v_r}{P} \leq \frac{v_r}{v_r + D}$ since $i_t > 0$ for some $t \leq r$. Hence $R \leq -\frac{h_r}{n_r} + \frac{1}{Q} \frac{\partial Q}{\partial i_r} + \frac{v_r}{v_r + D}$. Since u is sufficient, $u_r \geq \frac{n_r v_r}{D + v_r}$ and hence

$$\Sigma^* = \sum_{\{b=1, \dots, B: \omega_{b_r}=1\}} \kappa_b \geq \frac{n_r v_r}{v_r + D} - h_r.$$

When $m = 2$, $\gamma = 1$. When $m = 3$, either $\gamma = 1$ or $\gamma = 2$. It is easy to see that $\gamma = 1$ implies that Q is of the form $\kappa_1 s_1 + \kappa_2 s_2 + \kappa_3 s_3$, and that when $\gamma = 2$, Q is of the form $\kappa_1 s_1 s_2 + \kappa_2 s_1 s_3 + \kappa_3 s_2 s_3$. It can then be shown that $\frac{n_r}{Q} \frac{\partial Q}{\partial i_r} \leq -\Sigma^*$; this inequality is straightforward $\gamma = 1$ and follows by Lemma 4 (for example, when $r = 1$, with $x = \kappa_1, y = \kappa_2, z = \kappa_3, a = s_3, b = s_2, c = s_1$) when $\gamma = 2$. Thus $n_r R < -\Sigma^* \leq 0$. ■

Theorem 5. *Let $D \geq D_{crit}$, $m \leq 3$, and assume $u \in \mathbb{R}^m$ is sufficient. Then $(\vec{0}, \tilde{\alpha}(u))$ is a legal equilibrium.*

Proof: Let \vec{i} be any Inspectee strategy for which $i_1 \geq 1$. By Lemma 6, $\frac{\partial H(\vec{i}, \tilde{\alpha})}{\partial i_r} \leq 0$. Now let \vec{i} be any Inspectee strategy for which $i_t = 0$ for $t < r$ and $i_r \geq 1$. Again, $\frac{\partial H(\vec{i}, \tilde{\alpha})}{\partial i_r} \leq 0$. By induction on r , H is decreasing with respect to all \vec{i} other than $\vec{0}$. By Lemma 1, $H(\vec{e}_r, \alpha) \leq H(\vec{0}, \alpha)$ for $r = 1, \dots, m$. Thus H is decreasing with respect to some component for any Inspectee strategy $\vec{i} \neq \vec{0}$. Hence Inspectee can do no better than comply. When Inspectee complies, all Inspector payoffs are equal. Thus, Inspector can do no better than $\tilde{\alpha}$, and $(\vec{0}, \tilde{\alpha}(u))$ is a legal equilibrium. ■

4.4 The (n, k, m) Game when $k > 1$ and $m > 3$

We now provide a general sufficient condition for a legal equilibrium. To do so, we use a threshold value of D that is so large and an inspector mixed strategy that is so well-distributed over all sites that the expected number of inspections at any site, r , is never less than $\frac{n_r v_r}{v_r + D}$. Assume that $m \leq k$.

Definition 4. *For any $D > 0$, let $L(D)$ be the set of all values of $\ell = 1, 2, \dots, \mu$ such that Inspector strategy \vec{j}^ℓ satisfies $j_r^\ell \geq \frac{n_r v_r}{v_r + D}$ for all $r = 1, \dots, m$.*

If $L(D)$ is non-empty, then any mixture of (pure) inspector strategies in $L(D)$ has the property that the expected number of inspections at any level, r , is not less than $\frac{n_r v_r}{v_r + D}$.

Now let $D_* = \inf \left\{ D > 0 : \sum_{r=1}^m \left\lceil \frac{n_r v_r}{v_r + D} \right\rceil \leq k \right\}$. It is easy to verify that D_* is well-defined, and that $0 < D_{crit} \leq D_* < \infty$. Moreover, if $D > D_*$, then $L(D)$ is non-empty.

Definition 5. For fixed $D > D_*$, a star inspector strategy is an inspector mixed strategy, α^* , such that $\alpha_\ell^* = 0$ for all $\ell \notin L(D)$.

These ideas enable us to prove the next theorem.

Theorem 6. If $D > D_*$ and α^* is a star Inspector strategy, then $(\vec{0}, \alpha^*)$ is a Nash equilibrium.

Proof: Assume $D > D_*$ and let α^* be any star Inspector strategy. We use two lemmas to prove that Inspectee can do no better than comply by fixing any illegal Inspectee strategy $\vec{i} \neq \vec{0}$ and showing that $H(\vec{i}, \alpha^*) \leq H(\vec{0}, \alpha^*)$.

Lemma 7. Let r be the least integer such that $i_r \geq 1$. Then $H(\vec{i}, \alpha^*) \leq H(\vec{e}_r, \alpha^*)$.

Proof: Let $P(\vec{i}) = D + \sum_{\tau=1}^m i_\tau v_\tau$ so that, by (8), $H(\vec{i}, \alpha) = P(\vec{i}) \left(\sum_{\ell=1}^\mu \alpha_\ell \delta(\vec{i}, \vec{j}^\ell) \right)$. For any t , we formally differentiate $H(\vec{i}, \alpha^*)$ with respect to i_t making use of (12).

$$\begin{aligned} \frac{\partial}{\partial i_t} H(\vec{i}, \alpha^*) &= v_t \sum_{\ell \in L(D)} \alpha_\ell \delta(\vec{i}, \vec{j}^\ell) + P(\vec{i}) \sum_{\ell \in L(D)} \alpha_\ell \frac{\delta(\vec{i}, \vec{j}^\ell)}{S_{j_t^\ell}^{n_t}(i_t)} \frac{\partial S_{j_t^\ell}^{n_t}(i_t)}{\partial i_t} \\ &\leq v_t \sum_{\ell \in L(D)} \alpha_\ell \delta(\vec{i}, \vec{j}^\ell) - P(\vec{i}) \sum_{\ell \in L(D)} \alpha_\ell \delta(\vec{i}, \vec{j}^\ell) \frac{j_t^\ell}{n_t} \\ &= \sum_{\ell \in L(D)} \alpha_\ell \delta(\vec{i}, \vec{j}^\ell) \left(v_t - P(\vec{i}) \frac{j_t^\ell}{n_t} \right), \end{aligned}$$

by Lemma 5. Clearly, $\delta(\vec{i}, \vec{j}^\ell) \alpha_\ell \geq 0$. Because $\ell \in L(D)$, we have that $j_t^\ell \geq \frac{n_t v_t}{v_t + D}$, so that $j_t^\ell \geq \frac{n_t v_t}{P(\vec{i})}$ for $t \geq r$, with equality iff $t = r$ and $i_r = 1$. Because $j_t^\ell \geq \frac{n_t v_t}{P(\vec{i})}$ is equivalent to $v_t - P(\vec{i}) \frac{j_t^\ell}{n_t} \leq 0$, it follows that $\frac{\partial}{\partial i_t} H(\vec{i}, \alpha^*) \leq 0$, i.e. H is monotonically decreasing in i_t for $i_t \geq 0$ whenever $t > r$, and H is monotonically decreasing in i_r if $i_r \geq 1$. By hypothesis, $i_t = 0$ for $t = 1, 2, \dots, r-1$, and the lemma follows. ■

Lemma 8. $H(\vec{e}_r, \alpha^*) \leq H(\vec{0}, \alpha^*)$.

Proof: Because α^* is a star Inspector strategy,

$$\sum_{\ell=1}^\mu \alpha_\ell j_r^\ell = \sum_{\ell \in L(D)} \alpha_\ell j_r^\ell \geq \sum_{\ell \in L(D)} \alpha_\ell \frac{n_r v_r}{v_r + D} = \frac{n_r v_r}{v_r + D} \sum_{\ell \in L(D)} \alpha_\ell = \frac{n_r v_r}{v_r + D}.$$

By Lemma 1, this inequality implies that $H(\vec{e}_r, \alpha^*) \leq H(\vec{0}, \alpha^*)$, completing the proof. ■

Proof of Theorem 6: Given any inspectee strategy $\vec{i} \neq \vec{0}$, Lemma 7 implies that $H(\vec{i}, \alpha^*) \leq H(\vec{e}_r, \alpha^*)$ for some r and Lemma 8 implies that $H(\vec{e}_r, \alpha^*) \leq H(\vec{0}, \alpha^*)$. Therefore, in response to α^* , Inspectee can do no better than comply. ■

We know that no legal equilibria can occur for $D < D_{crit}$, but that all equilibria are legal for $D > D_*$. What about values of D satisfying $D_{crit} \leq D \leq D_*$? We conjecture that legal equilibria occur whenever $D \geq D_{crit}$. This conjecture is true in the case $m = 1$, which we discussed earlier, and in the cases $m = 2$ and $m = 3$, which are covered in the previous section. As long as our conjecture remains unproven, calculation of D_* provides values of D that are guaranteed to be large enough to deter violations. But if our conjecture could be shown to be true in general, D_* would have no other interest or practical value, so far as we are aware.

It seems difficult to find an upper bound for the gap between D_{crit} and D_* , which can be quite large. Consideration of the behavior of functions of the form $f(D) = \frac{nv}{v+D}$ suggests that this gap tends to decrease as k increases. We offer two illustrative examples, which both assume $m = 4, v_1 = 4, v_2 = 3, v_3 = 2$, and $v_4 = 1$. In the first example, $n_1 = 400, n_2 = 350, n_3 = 300, n_4 = 250$, and $k = 12$, which produces $D_{crit} = 288.52$ and $D_* = 316$; D_* is roughly 9.5% larger than D_{crit} . In the second example, $n_1 = 6, n_2 = 6, n_3 = 5, n_4 = 4$, and $k = 4$, yielding $D_{crit} = 10.959$ and $D_* = 20$, roughly 82.5% larger. The second example has the largest percentage gap between D_{crit} and D_* that we have encountered.

5 Conclusions

Our analysis demonstrates that there is good reason to believe that the threshold level of inspection in an (n, k, m) inspection game is equal to (or perhaps slightly larger than) D_{crit} , the unique solution of (11),

$$\sum_{r=1}^m \frac{n_r v_r}{v_r + D} = k.$$

The relation of D_{crit} with the parameters describing the site distribution, n_1, n_2, \dots, n_m , Inspectee's site values, v_1, v_2, \dots, v_m , and the number of inspections, k , is complex, but some observations can be made.

Consider the inequality

$$\sum_{\ell=1}^{\mu} \alpha_{\ell} J_r^{\ell} \geq \frac{n_r v_r}{v_r + D}.$$

The left-hand side represents the expected number of inspections at level r . On the right-hand side, $v_r + D = v_r - (-D)$ measures the range of outcomes possible for an inspectee at

one site, so $\frac{v_r}{v_r+D}$ is the unitless relative benefit to inspectee of a single successful violation. If $D = 0$, this fraction assumes the value one; if $D > 0$, it is less than one. We think of $\frac{v_r}{v_r+D}$ as a “discount factor” that shrinks n_r sites down to $\frac{n_r v_r}{v_r+D}$, which we regard as the effective number of sites at level r , the reduction being due to the presence of the possible penalty, D . Thus Theorem 5 tells us that if the expected number of inspections exceeds the effective number of sites at every level and $m \leq 3$, legal equilibria exist. One plausible interpretation of our results is this: While it would rarely be possible for inspector to visit every site, an inspector can make a “virtual visit” to every site if D is sufficiently large and site selection is appropriately randomized. A regime of such virtual visits shifts an inspectee’s incentives in favor of compliance.

We assumed above that $m \leq k \leq \min\{n_r : r = 1, \dots, m\}$. Assuming that none of these assumptions is violated, two coarse approximations give some information about the relation of D_{crit} with the n ’s. If v_1 is much larger than v_2, v_3, \dots, v_m , then $D_{crit} \approx \frac{n_1 v_1}{k} - v_1$. If $v_1 \approx v_2 \approx \dots \approx v_m = v$, then $D_{crit} \approx \sum n_r \frac{v}{k} - v$. These calculations suggest that adding another inspectable site will increase the threshold penalty for deterrence by roughly the value of the site divided by k , the number of inspections.

To learn something about the relation of the threshold penalty to site values, we carry out the same two approximations. If the values of all sites are negligible in comparison to v_1 , we find $D_{crit} \approx (\frac{n_1}{k} - 1) v_1$. (Note that $k \leq n_1$ by the assumption above.) If the values of all sites are approximately equal to v , we obtain $D_{crit} \approx (\frac{1}{k} \sum n_r - 1) v$. Both cases suggest that threshold penalty is a positive multiple of value. This inference almost certainly overstates the dependence, because in both approximations the increase in value applies to all non-negligible sites. We expect that the increase in threshold penalty consequent on the increase in value of all sites on any one level is less than proportionate.

A major issue for most arms-control treaties and regimes is the amount of inspection—the extent to which inspectors can intrude on an inspectee’s territory and facilities, possibly disrupting activities and programs and possibly gathering military or commercial intelligence. If the (n, k) and (n, k, m) inspection games are taken as models of an inspection regime, the amount of inspection is represented by the parameter k . Since Inspectee can be expected to insist that the value of k be as low as possible, an assessment of the relation between k and the threshold penalty level for deterrence — which we take to be D_{crit} , the solution of (11) — is of interest.

Since (11) defines $D = D_{crit}$ implicitly as a function of k , we can differentiate implicitly to obtain

$$\frac{\partial D}{\partial k} = -\frac{1}{\sum_{r=1}^m \frac{n_r v_r}{(v_r+D)^2}}. \quad (15)$$

Unfortunately, it is difficult to gain any insight into the relation of D and k directly from (15), except to note that the derivative is negative, so increases in k can be expected to lead to decreases in D .

Instead, we resort to the two coarse approximations used above. If the values of all sites are negligible in comparison to v_1 , we find $D_{crit} \approx n_1 v_1 \left(\frac{1}{k}\right) - v_1$. If the values of all sites are approximately equal to v , we obtain $D_{crit} \approx \sum n_r v \left(\frac{1}{k}\right) - v$. In both cases, we have the suggestion that threshold penalty is a positive linear function of $\frac{1}{k}$. It follows that the change in D induced by one extra inspection is inversely proportional to k^2 . These observations suggest that if a treaty can be negotiated with a relatively large penalty, then the number of inspections can be small, in which case adding an inspection would cause a large reduction in the threshold penalty. On the other hand, if because of other factors the penalty is inevitably small, then the number of inspections must be large, and additional inspections will have a relatively small effect on the threshold penalty. In particular, small increases in the number of inspections will not likely improve security significantly.

Our objective in this paper has been to use game models to understand better how underlying characteristics affect optimal behavior in multi-site inspection processes. While there are many open questions, including our conjecture about the range $D_{crit} < D < D_*$, we have uncovered some facts about how optimal inspection strategies are related to the pattern of site values, and about how the number of inspections allowed by a regime is related to its ability to deter violations. Our terminology and descriptions may have reflected an arms-control viewpoint, but the implications of our results are clear for other inspection regimes, such as customs, weights and measures, food and drug verification, and environmental management. The number of regulations to be enforced can surely be expected to increase in the future, so finding ways to make enforcement more effective and more efficient will benefit everyone.

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