Separable Discrete Preferences

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Abstract

An ordering of multidimensional alternatives is separable on a set of dimensions if fixing values on the complementary dimensions always produces the same induced ordering. Most often, studies of separability assume continuous alternative spaces; as we show, separability has different properties when alternative spaces are discrete. For instance, two well-known theorems of Gorman—that common set operations preserve separability and that separable preferences are additive—fail for binary alternative spaces. Intersection is the only set operation that preserves separability. For binary alternative spaces, separability is equivalent to additivity if and only if there are four or fewer dimensions.

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1 Introduction

When a group decides on several issues simultaneously, interdependence within individuals' preferences can lead to outcomes that are unsatisfactory or even paradoxical. A classic example is the "paradox of multiple elections" observed by Brams, Kilgour, and Zwicker (1998), in which every voter disagrees with the outcome on at least one question. Lacy and Niou (2000) demonstrate that the outcome can even be the last choice of every voter.

The notion of separability has been introduced in economics, political science, and operations research to describe preferences that do not exhibit such interdependencies. For example, if a voter in a simultaneous referendum with two issues has a separable preference, then whether the voter supports or opposes each proposition does not depend on how the other proposition is decided. But if the referendum contains two bond proposals that draw funds from the same tax base, then the voter's preference on one proposal might well depend on what decision is made on the other. If so, the voter's preference is nonseparable.

While the concept of separability is the same across the disciplines, the topology of the alternative sets is usually different. In economics, and in some applications in political science (e.g. spatial models), it is natural to assume that alternative sets are continuous. But in multiple elections they are naturally discrete; in simultaneous referenda, they are binary.

Up to now, separability has been studied mainly in the context of continuous alternative spaces. In this paper, we will examine the properties of separability when alternative spaces are discrete. Separability is important in this context; for example, in the strategic voting and social choice literature, separability has been shown to be crucial to the existence of nondictatorial strategyproof social choice functions. Well-known recent works referring to separability include Barberà, Sonnenschein, and Zhou (1991), Barberà, Massó, and Neme (1997), LeBreton and Sen (1999), and Chatterji, Sen, and Aswal (2003).

Section 2 presents the model underlying this article, and then Section 3 formalizes separability and derives some basic properties. Section 4 studies separability when alternative spaces are discrete, and shows that finite intersections are the only common set operations that preserve separability, contradicting the conclusion of the well-known theorem of Gorman (1968). Section 5 formalizes the concept of additivity and shows that Gorman's (1968) second theorem—that separable preferences are additive—also fails in the discrete case in five or more dimensions. In Section

6 some necessary conditions for separability when alternative spaces are binary are developed, and separability and additivity are then proven equivalent when there are four or fewer alternatives. A taxonomy of separable preferences over such spaces is also presented. Section 7 summarizes our results and suggests some avenues for future work.

2 Multidimensional Preferences

To allow for a range of alternative sets, we adopt a general multiple-criteria decision model. We begin with a *set of criteria*, Q, upon which we impose no particular structure. For each *criterion* $q \in Q$, let X_q be a set called the *alternative set for criterion* q. Then an *alternative* (or *outcome*) is an element of the Cartesian product[1]

$$X_Q = \prod_{q \in Q} X_q.$$

Let $x \in X_Q$ and $S \subseteq Q$. Denote by x_S the projection of x onto S, so that $x_S \in X_S$ where $X_S = \prod_{q \in S} X_q$. We assume that criteria can be reordered as convenient, so that if $\mathcal{P} = \{S_1, S_2, \ldots, S_n\}$ is a partition of Q, then we can write $x = (x_{S_1}, x_{S_2}, \ldots, x_{S_n})$.[2] In particular, we use $-S$ to denote the complement of S in Q, and write $x = (x_S, x_{-S})$.

We represent a preference ordering by a binary relation $\succeq$ on X_Q. Often, $\succeq$ is assumed to be a partial order, a weak order, or a linear order (see Fishburn (1970), for example). For now, we impose no such restrictions, but for clarity, our examples will usually involve linear orders. Given a relation $\succeq$ on X_Q, let $\succ$ be the relation defined by $x \succ y$ iff $x \succeq y$ and $y \not\succeq x$.

Example 2.1. Multidimensional alternatives appear in the following political science contexts:

- **Spatial Model**. In a typical spatial model, each X_q is an interval of real numbers, or the entire real number line, representing possible policy positions on issue q. See, for example, Enelow and Hinich (1984).

- **Multiple Elections**. To model simultaneous, multi-candidate, single-winner elections, let

[1] In most applications in political science and economics, every alternative in X_Q is available for selection, whereas in multiple-criteria decision applications, the available alternatives often constitute only a subset of X_Q, perhaps a very small subset. For our purposes, this distinction is irrelevant.

[2] We allow some elements of $\mathcal{P}$ to be empty, in which case $x = (x_{S_1}, x_{S_2}, \ldots, x_{S_n})$ is taken to mean $x = (x_{S_{i_1}}, x_{S_{i_2}}, \ldots, x_{S_{i_m}})$, where $\{i_k : k = 1, \ldots, m\}$ is the set of indices of nonempty elements of $\mathcal{P}$.

X_q represent the candidates for office q. See, for example, Brams, Kilgour, and Zwicker (1998) or Schwartz (1977).

- **Simultaneous Referenda**. To model simultaneous referenda on the questions in Q, let $X_q = \{0, 1\}$ to represent No (0) and Yes (1) on each question. See, for example, Brams, Kilgour, and Zwicker (1998), Hodge and Bradley (1998), Hodge (2002), Kilgour (1997), and Lacy and Niou (2000).

In the first example, each X_q is infinite (in fact, continuous and arc-connected), whereas in the latter two examples, each X_q is finite. Of course, a referendum is an election with two candidates, so the third example is a special case of the second.

3 Separability

Roughly, a preference ordering is separable on a subset S of the criteria if the ordering of alternatives within S never depends on the choice of alternatives outside S. Formally,

Definition 3.1. Let $S \subset Q$. Then S is *$\succeq$-separable*, or *separable with respect to $\succeq$*, iff, for all $x_S, y_S \in X_S$,

$$(x_S, u_{-S}) \succeq (y_S, u_{-S}) \text{ for some } u_{-S} \in X_{-S}$$

implies

$$(x_S, v_{-S}) \succeq (y_S, v_{-S}) \text{ for all } v_{-S} \in X_{-S}.$$

A criterion $q \in Q$ is said to be $\succeq$-separable iff $\{q\}$ is $\succeq$-separable. The relation $\succeq$ is said to be *separable* iff each nonempty $S \subset Q$ is $\succeq$-separable.

In most cases, we are interested only in nonempty, proper subsets of criteria, so for convenience we augment Definition 3.1 with the convention that both Q and $\emptyset$ are $\succeq$-separable. If a set S is $\succeq$-separable, then we sometimes say that $\succeq$ is separable on S.

Our formulation of separability is based on Yu's (1985) definition. According to Gorman (1968), the concept of separability originated in Leontief's early work. Keeney and Raiffa (1976) use the phrase "preference independence with respect to the complement" to mean $\succeq$-separability. Schwartz (1977), who seems to have been the first to consider separability in the context of multiple elections, suggested the term "noncomplementary."

Example 3.2. Suppose that two questions are to be decided in simultaneous referenda, so that $Q = \{q_1, q_2\}$ and $X_{q_1} = X_{q_2} = \{0, 1\}$. We denote the four alternatives, or possible outcomes, by 11, 10, 01, and 00 (where, for example, 10 represents Yes on q_1 and No on q_2). Assuming that preference orderings over X_Q are strict, we write them vertically, from the most preferred alternative at the top to the least preferred at the bottom. We call this representation a *preference matrix*, and identify the relation $\succ$ with the matrix R that represents it.

Consider the three preference matrices

$$R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad R_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to verify that R_1 is separable with respect to q_2 but not with respect to q_1, R_2 is separable on both questions, and R_3 not separable on either question. To illustrate, R_1 is separable on q_2 because 1 precedes 0 on q_2, whether the outcome on q_1 is 1 or 0. On the other hand, R_1 is not separable on q_1, because 1 precedes 0 on q_1 if the outcome on q_2 is 0, whereas 0 precedes 1 on q_1 if the outcome on q_2 is 1.

Now consider the following preference matrices, which might be held by a voter facing simultaneous referenda on three questions:

$$R_4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad R_5 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that R_4 is separable on each question and that the subsets $\{q_1, q_2\}$ and $\{q_2, q_3\}$ are R_4-separable, but the subset $\{q_1, q_3\}$ is not. For example, notice that 011 is preferred to 110 but 100 is preferred to 001, which indicates that 01 is preferred to 10 on $\{q_1, q_3\}$ when $x_{q_2} = 1$, but that 10 is preferred to 01 on $\{q_1, q_3\}$ when $x_{q_2} = 0$. For R_5, it can be verified that $\{q_1\}$ and $\{q_2, q_3\}$ are separable, but that $\{q_2\}$, $\{q_3\}$, $\{q_1, q_2\}$, and $\{q_1, q_3\}$ are not.

When a subset S of Q is not separable with respect to an ordering, it is natural to ask which criteria outside S affect the induced ordering on X_S. In fact, it is more useful to identify those criteria that do not influence the induced ordering on X_S. (See Hodge and Bradley (1998), and Hodge (2002).)

Definition 3.3. Let S and T be disjoint subsets of Q. Then T is said to be *non-influential on S with respect to* $\succeq$ iff, for all x_S, $y_S \in X_S$,

$$(x_S, u_T, w_{-(S\cup T)}) \succeq (y_S, u_T, w_{-(S\cup T)}) \text{ for some } (u_T, w_{-(S\cup T)}) \in X_{-S}$$

implies

$$(x_S, v_T, w_{-(S\cup T)}) \succeq (y_S, v_T, w_{-(S\cup T)}) \text{ for all } v_T \in X_T.$$

For instance, in Example 3.2, it can be verified that $\{q_1\}$ is not only separable with respect to R_5, it is also non-influential on each of $\{q_2\}$, $\{q_3\}$, and $\{q_2, q_3\}$ with respect to R_5. Since $\{q_1\}$ is non-influential on every subset of $\{q_2, q_3\} = Q - \{q_1\}$, we say that $\{q_1\}$ is *non-influential* (with respect to R_5). With this definition, it can be shown that a set $S \subseteq Q$ is separable iff $-S$ is non-influential.

4 Separability and Set Operations

It is natural to ask which set operations preserve separability. Before discussing a well-known theorem on this subject, we introduce a few definitions. Let S, $T \subseteq Q$. The *symmetric difference* of S and T is $S \bigtriangleup T = (S - T) \cup (T - S)$. S and T *overlap* iff both $S \cap T$ and $S \bigtriangleup T$ are nonempty. S is *essential* with respect to $\succeq$ iff there exist $u_{-S} \in X_{-S}$ and x_S, $y_S \in X_S$ such that $(x_S, u_{-S}) \succ (y_S, u_{-S})$, and *strictly essential* with respect to $\succeq$ iff, for all $u_{-S} \in X_{-S}$, there exist x_S, $y_S \in X_S$ such that $(x_S, u_{-S}) \succ (y_S, u_{-S})$.

Theorem 4.1 (Gorman, 1968). *Suppose $\succeq$ is a continuous, complete relation on X_Q, and that X_q is topologically separable and arc-connected for each $q \in Q$.*[3] *Suppose further that $S, T \subseteq Q$*

[3]A relation $\succeq$ is *continuous* if the sets $\{y \in X_Q : x \succeq y\}$ and $\{y \in X_Q : y \succeq x\}$ are closed for each $x \in X_Q$. It is *complete* if, for all x, $y \in X_Q$, either $x \succeq y$ or $y \succeq x$. A set X is *topologically separable* if it contains a countable

overlap, that either $S - T$ or $T - S$ is strictly essential, and that $\{q\}$ is essential for each $q \in Q$. If S and T are $\succeq$-separable, then all of $S \cup T$, $S \cap T$, $S - T$, $T - S$, and $S \triangle T$ are $\succeq$-separable and strictly essential.

Gorman's Theorem concludes that separability is preserved under unions, intersections, and symmetric differences. Some of Gorman's hypotheses are quite innocuous. For example, if $\succeq$ is a strict order (as are all the orderings in Example 3.2), then every nonempty, proper subset of Q is strictly essential. The requirement of arc-connectedness, however, is a severe restriction. In voting contexts, for example, the sets X_q are discrete, and thus cannot be arc-connected. Thus Theorem 4.1 does not apply to orderings such as those in Example 3.2. In fact, the conclusions of Theorem 4.1 are false for the ordering R_4, as shown by the following observations:

1. $S = \{1\}$ and $T = \{3\}$ are separable, but $S \cup T = \{1, 3\}$ is not.
2. $S = \{1, 2\}$ and $T = \{2, 3\}$ are separable, but $S \triangle T = \{1, 3\}$ is not.
3. $S = \{1, 2, 3\}$ and $T = \{2\}$ are separable, but $S - T = \{1, 3\}$ is not.

Together, R_4 and R_5 demonstrate that separability is not inherited from either subsets or supersets—in other words, a set may not be separable even when all of its subsets, or all of its supersets, are separable. These observations indicate that the preservation of separability under set operations depends in an essential way on the topological nature of the alternative sets. The next result appears to be the only general structural relationship satisfied by separability.

Proposition 4.2. *If S_1 and S_2 are $\succeq$-separable, then $S_1 \cap S_2$ is $\succeq$-separable.*

Proof. Suppose $S_1, S_2 \subseteq Q$ and let $S = S_1 \cap S_2$, $T_1 = S_1 - S$, $T_2 = S_2 - S$, and $T = -(S_1 \cup S_2)$. Suppose that $(x_S, u_{T_1}, u_{T_2}, u_T) \succeq (y_S, u_{T_1}, u_{T_2}, u_T)$ for some $u \in X_{-S}$. Choose $v \in X_{-S}$. Since S_1 is separable and $(v_{T_2}, v_T) \in X_{-S_1}$, it follows that $(x_S, u_{T_1}, v_{T_2}, v_T) \succeq (y_S, u_{T_1}, v_{T_2}, v_T)$. But S_2 is separable and $(v_{T_1}, v_T) \in X_{-S_2}$, which implies that $(x_S, v_{T_1}, v_{T_2}, v_T) \succeq (y_S, v_{T_1}, v_{T_2}, v_T)$. Because v is arbitrary, it follows that $S = S_1 \cap S_2$ is separable. $\square$

By standard methods, Proposition 4.2 can be generalized to show that the intersection of any finite collection of separable sets is separable. A consequence is that, if Q is finite and $S \subseteq Q$, then

dense subset, and is *arc-connected* if, for all $x, y \in X$, there exists a continuous function $\phi : [0, 1] \to X$ such that $\phi(0) = x$ and $\phi(1) = y$.

S is $\succeq$-separable if $Q - \{q\}$ is $\succeq$-separable for every $q \notin S$, since

$$S = \bigcap_{q \notin S} (Q - \{q\}).$$

Moreover, the relation $\succeq$ is separable if $Q - \{q\}$ is $\succeq$-separable for every $q \in Q$. In fact, this sufficient condition, which, by the definition of separability, is also necessary, provides a useful method to detect separability of a voter's preference in a multiple referendum. If the referendum has n issues, preference separability (which provides information about preferences on all 2^n subsets of Q) can be demonstrated by the answers to n Yes or No questions.

The possible generalization of Proposition 4.2 to infinite collections of sets remains an open question.

5 Separability and Additivity

Henceforth we assume that the set of criteria, Q, is finite.

Definition 5.1. Let $\succeq$ be a weak order on X_Q. A *value function* for $\succeq$ is a function $v : X_Q \to \mathbb{R}$ such that for all x, $y \in X_Q$, $x \succeq y$ iff $v(x) \geq v(y)$. The relation $\succeq$ is *additive* iff for each $q \in Q$ there exists a *criterion value* $v_q : X_q \to \mathbb{R}$ such that $v(x) = \sum_{q \in Q} v_q(x_q)$ is a value function for $\succeq$.

If X_Q is finite, it is obvious that any ordering on X_Q has a value function. But not all such orderings are additive. For instance, of the five linear orders in Example 3.2, only R_2 is additive. In fact, R_2 has a particularly simple value function, $v(x_{q_1}, x_{q_2}) = -x_{q_1} + 2x_{q_2}$.

If each X_q can be identified with a subset of $\mathbb{R}$, an additive ordering may have a value function for which all criterion values are linear. A *linear value function* is a value function of the form

$$v(x) = \sum_{q \in Q} c_q x_q,$$

where $c_q \in \mathbb{R}$ for each $q \in Q$. If a linear value function exists, $\succeq$ is said to be *linearly additive.* As noted above, R_2 is linearly additive.

Linear value functions are particularly convenient. We will show next that additivity and linear additivity are equivalent when alternatives are binary. Our proof is based on a simple lemma.

Lemma 5.2. *Let $\succeq_1$ and $\succeq_2$ be additive orders on X_Q with respective value functions v and w. If $v - w$ is constant, then $\succeq_1 = \succeq_2$.*

Proof. Let $c \in \mathbb{R}$ have the property that $v(x) - w(x) = c$ for all $x \in X_Q$. Then

$$v(x_1) \geq v(x_2) \iff w(x_1) + c \geq w(x_2) + c \iff w(x_1) \geq w(x_2),$$

so $x_1 \succeq_1 x_2$ iff $x_1 \succeq_2 x_2$. $\square$

Proposition 5.3. *Let $Q = \{1, 2, \ldots, n\}$ and let $X_q = \{a_q, b_q\} \subset \mathbb{R}$, where $a_q \neq b_q$ for each $q \in Q$. Suppose that $\succeq$ is a weak order on X_Q. Then $\succeq$ is additive iff $\succeq$ is linearly additive.*

Proof. Assume that $\succeq$ is additive with value function $v(x) = \sum_{q=1}^{n} v_q(x_q)$. Because $X_q = \{a_q, b_q\}$, we can write

$$v_q(x_q) = \frac{v_q(b_q) - v_q(a_q)}{b_q - a_q}(x_q - a_q) + v_q(a_q),$$

for each $x_q \in X_q$ and each $q \in Q$. Now let

$$c_q = \frac{v_q(b_q) - v_q(a_q)}{b_q - a_q}.$$

Then

$$v(x) = \sum_{q=1}^{n} c_q x_q + \sum_{q=1}^{n} (v_q(a_q) - c_q a_q).$$

The latter sum is a constant, so Lemma 5.2 implies that $w(x) = \sum_{q=1}^{n} c_q x_q$ is a value function for $\succeq$, proving that $\succeq$ is linearly additive. The reverse implication is immediate. $\square$

To show that the hypothesis $|X_q| \leq 2$ is essential for Proposition 5.3, take $Q = \{1, 2\}$, $X_1 = X_2 = \{-1, 2, -3\}$, and let $\succeq$ represent the additive order with value function $v(x) = x_1 + x_2^2$. Denote by $\sim$ the indifference relation corresponding to $\succeq$.[4] It can be verified readily that $(-3, -3) \sim (2, 2)$ and $(-1, 2) \sim (2, -1)$, which implies that a linear value function $v^*(x) = c_1 x_1 + c_2 x_2$ for $\succeq$ would satisfy

$$-3c_1 - 3c_2 = v^*((-3, -3)) = v^*((2, 2)) = 2c_1 + 2c_2$$

$$-c_1 + 2c_2 = v^*((-1, 2)) = v^*((2, -1)) = 2c_1 - c_2.$$

Simple manipulation shows that the unique solution to this system of equations is $c_1 = c_2 = 0$. But then $x \sim y$ for all $x, y \in X_Q$, which is clearly not the case. This contradiction shows that the additive ordering $\succeq$ is not linearly additive.

[4] The *indifference relation* $\sim$ corresponding to a weak order $\succeq$ is the relation defined by $x \sim y$ iff $x \succeq y$ and $y \succeq x$. If $\succeq$ is an additive weak order with value function v, then $x \sim y$ iff $v(x) = v(y)$.

Is it a coincidence that the only separable ordering in Example 3.2 is also the only additive ordering? The connections between separability and additivity have been studied extensively; we summarize two important results below. The first has appeared in several locations, for example Yu (1985). The second is due to Gorman (1968).

Proposition 5.4. *Let $\succeq$ be an additive order on X_Q. Then $\succeq$ is separable.*

Theorem 5.5. *Suppose that, for each $q \in Q$, $\{q\}$ is strictly essential and X_q is topologically separable and arc-connected. Suppose also that $\succeq$ is a continuous weak order on X_Q. Then $\succeq$ is additive if and only if $\succeq$ is separable.*

Note the hypotheses of Theorem 5.5, like those of Theorem 4.1, exclude the discrete case. We now demonstrate that just like Theorem 4.1, the conclusion of Theorem 5.5 is false when the criterion alternative sets are discrete.

Example 5.6. Let $Q = \{1,2,3,4,5\}$, where $X_q = \{0,1\}$ for each $q \in Q$. Let $\succ$ be the linear order on X_Q given by the 32×5 preference matrix P, of which the first 16 rows constitute the 16×5 matrix P_U and the last 16 rows 16×5 matrix P_L, as follows:

$$P_U = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}, \quad P_L = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The ordering $\succ$, represented by P, was developed from an example due to Kraft, Pratt, and Seidenberg (1959). Using our remark following Proposition 4.2, it is not difficult to verify that $\succ$ is separable.

But $\succ$ is not additive. To see this, suppose first that $\succ$ has a linear value function $v(x) = \sum_{q=1}^{5} c_q x_q$. From P, we have that

$$\begin{aligned}(0,1,1,0,1) &\succ (1,0,0,1,0)\\ (1,0,0,0,1) &\succ (0,1,1,0,0)\\ (0,0,1,1,0) &\succ (0,1,0,0,1)\\ (0,1,0,0,0) &\succ (0,0,1,0,1).\end{aligned}$$

It follows that

$$\begin{aligned}c_2 + c_3 + c_5 &> c_1 + c_4\\ c_1 + c_5 &> c_2 + c_3\\ c_3 + c_4 &> c_2 + c_5\\ c_2 &> c_3 + c_5.\end{aligned}$$

Adding the first three inequalities yields $c_3 + c_5 > c_2$, which contradicts the fourth inequality. It follows that $\succ$ cannot be linearly additive. By Proposition 5.3, $\succ$ is not additive.

It is easy to modify Example 5.6 to show that there exists a separable but nonadditive order on n criteria whenever $n \geq 5$. Specifically, let $Q = \{1, 2, 3, \ldots, n\}$ and let $\succ$ be any linear order on X_Q such that

$$\begin{aligned}(0,1,1,0,1,0,\ldots,0) &\succ (1,0,0,1,0,0,\ldots,0)\\ (1,0,0,0,1,0,\ldots,0) &\succ (0,1,1,0,0,0,\ldots,0)\\ (0,0,1,1,0,0,\ldots,0) &\succ (0,1,0,0,1,0,\ldots,0)\\ (0,1,0,0,0,0,\ldots,0) &\succ (0,0,1,0,1,0,\ldots,0).\end{aligned}$$

Such an order can be constructed by "crossing" P with any separable order on criteria $6, 7, \ldots, n$. Now the argument above applies, and shows that $\succ$ is not additive.

It is no coincidence that we have been able to establish the non-equivalence of separability and additivity for $|Q| \geq 5$ only. Example 5.6 is in fact the smallest possible counterexample to the conclusion of Theorem 5.5 in the binary case. The proof that if $|Q| \leq 4$ all separable preferences on binary alternative sets are additive is the main content of the next section.

6 Separable Preferences on Binary Alternative Sets

We now assume that $Q = \{1, 2, \ldots, n\}$ and that $X_q = \{0, 1\}$ for each $q \in Q$. We identify properties of separable linear orders, i.e. preference matrices, on X_Q, use the resulting characterization to exhibit all such orders for $n = 2$, 3, or 4, and then demonstrate that the converse to Proposition 5.4 holds when $n \leq 4$. Combined with Example 5.6, this result shows that separable preferences on n binary alternative sets are additive iff $n \leq 4$.

First, we standardize the problem. Let P be a preference matrix representing a linear order $\succ$ on X_Q. The first row of P is an outcome called the *leader* of P and $\succ$. The order $\succ$ and the preference matrix P are said to be *normalized* if the leader is $11\ldots1$. They are said to be *canonically ordered* if the rows of P satisfy

$$10\ldots00 \succ 01\ldots00 \succ \ldots \succ 00\ldots10 \succ 00\ldots01.$$

Any preference matrix can be obtained from a normalized, canonically ordered preference matrix by permuting columns and taking bitwise complements of columns as appropriate. Without loss of generality, we henceforth assume that all preference matrices are normalized and canonically ordered.

Any alternative in X_Q can be viewed as a subset of Q, namely the set of criteria on which the value of the alternative is 1. For instance, if $n = 5$, the alternative 11010 corresponds to the subset $\{1, 2, 4\}$. A preference matrix thus corresponds to a linear ordering of $\mathcal{P}(Q)$, the set of all subsets of Q. Note that we have assumed that the leader of this ordering is Q itself, and that $\{1\} \succ \{2\} \succ \ldots \succ \{n\}$.

We now obtain some necessary properties of separable preference matrices, which we will use later to characterize separability. For an alternative $x \in X_Q$, let $\overline{x}$ denote the bitwise complement of x; for example, if $x = (1, 0, 1)$, then $\overline{x} = (0, 1, 0)$. Obviously, $\overline{\overline{x}} = x$.

A relation $\succ$ is said to have the *mirror property* iff, for all distinct x, $y \in X_Q$, $x \succ y$ implies $\overline{y} \succ \overline{x}$.

Proposition 6.1. *If $\succ$ is separable, then $\succ$ has the mirror property.*

Proof. Let x, $y \in X_Q$ and suppose that $x \succ y$. If $x = \overline{y}$, then $\overline{x} = y$, so $\overline{y} = x \succ y = \overline{x}$, as required. Now suppose that $x \neq \overline{y}$ and let $S = \{q \in Q \mid x_q = y_q\}$. By assumption, S is a nonempty proper subset of Q. Also notice that $x_S = y_S$ and $x_{-S} = \overline{y_{-S}}$. Thus,

$$(y_S, \overline{y_{-S}}) = (x_S, x_{-S}) = x \succ y = (y_S, y_{-S}).$$

The separability of $\succ$ now implies that

$$\overline{y} = (\overline{y_S}, \overline{y_{-S}}) \succ (\overline{y_S}, y_{-S}) = (\overline{x_S}, \overline{x_{-S}}) = \overline{x}. \qquad \square$$

Proposition 6.2. *If $\succ$ is separable, then the $(2^n - i + 1)^{st}$ row of P is the bitwise complement of the i^{th} row of P.*

Proof. Without loss of generality, assume that $i \leq 2^{n-1}$. For all $1 \leq j \leq 2^n$, let $x_j \in X_Q$ be the outcome corresponding to row j. Then $x_j \succ x_i$ for all $j < i$ and $x_i \succ x_j$ for all $j > i$. But by Proposition 6.1, $\overline{x}_i \succ \overline{x}_j$ for all $j < i$ and $\overline{x}_j \succ \overline{x}_i$ for all $j > i$. Since there are exactly $2^n - i + 1$ values of j for which $j > i$, it follows that $\overline{x}_i = x_{2^n - i+1}$, as claimed. $\square$

According to Proposition 6.2, in a separable preference matrix the i^{th} row from the top is the bitwise complement of the i^{th} row from the bottom. Of the matrices $R_1, \ldots, R_5$ in Example 3.2, only R_2 has this property. Recall that R_2 is separable.

Now we construct an important partial order on $\mathcal{P}(Q)$. Then we show that the separable orders must constitute a certain class of linear extensions of this partial order.

Let $S = \{s_1, s_2, \ldots, s_p\}$ and $T = \{t_1, t_2, \ldots, t_p\}$ be subsets of Q with $|S| = |T|$, where $s_1 < s_2 < \ldots < s_p$ and $t_1 < t_2 < \ldots < t_p$. Suppose that, for some k, $s_i = t_i$ if and only if $i \neq k$. Then S is said to *immediately dominate* T if $s_k = t_k - 1$. For example, $\{1, 2, 4, 6\}$ immediately dominates $\{1, 2, 5, 6\}$.

Let $\succeq_d$ be the relation on $\mathcal{P}(Q)$ defined by

$$S_1 \succeq_d S_2 \text{ if and only if } S_1 \supset S_2 \text{ and } |S_1| = |S_2| + 1, \text{ or } S_1 \text{ immediately dominates } S_2,$$

and define $\succeq_D$ to be the transitive closure[5] of $\succeq_d$. Figure 1 shows digraph representations of $\succeq_D$ for $n = 2, 3, 4$, and 5. The direction of each relation is downward.

[5]The transitive closure of immediate domination is also known as the *right shift ordering.* Taylor and Zwicker (1999) observe that this ordering has been rediscovered many times, and give many references.

Figure 1: Digraph representations of $\succeq_D$

It is evident from Figure 1 that $(\mathcal{P}(Q), \succeq_D)$ is a lattice with maximal element Q and minimal element $\emptyset$. Another noteworthy feature of this lattice is symmetry under complementation: if every subset is replaced by its complement and the relation reversed, the lattice is unchanged.

Next we show that, if a preference matrix P is separable, the corresponding subset order $\succ$ must be consistent with the partial order $\succeq_D$ on $\mathcal{P}(Q)$. In the next proposition, rows of P are denoted by lower case letters (r and s) and corresponding subsets of Q by upper case letters (R and S, respectively).

Proposition 6.3. *Suppose P is a separable preference matrix corresponding to the relation $\succ$ on X_Q. Let r and s be rows of P such that R and S are comparable with respect to $\succeq_D$. Then $R \succeq_D S$ implies $r \succ s$.*

Proof. First assume that $R \succeq_d S$, so that R and S are adjacent in $(\mathcal{P}(Q), \succeq_D)$. Either $R \supset S$ and $|R| = |S| + 1$, or $|R| = |S|$ and R immediately dominates S.

In the first case, there exists j such that $R = S \cup \{j\}$. It follows that $r_j = 1$, $s_j = 0$, and $r_i = s_i$ for $i \neq j$. Then $r \succ s$ by normalization and the separability of $\succ$.

In the second case, there exist j and $k = j + 1$ such that $R = \{j\} \cup T$ and $S = \{k\} \cup T$, where $T \subseteq Q - \{j, k\}$. Hence $r_j = 1$, $r_k = 0$, $s_j = 0$, $s_k = 1$, and $r_i = s_i$ for $i \notin \{j, k\}$. Then $r \succ s$ by canonical ordering and the separability of $\succ$.

If $R \succeq_D S$ but not $R \succeq_d S$, then there exist $R_1, R_2, \ldots, R_m$ such that $R = R_1 \succeq_d R_2 \succeq_d \ldots \succeq_d R_m = S$. By the arguments above, $r = r_1 \succ r_2 \succ \ldots \succ r_m = s$ as required. $\square$

Proposition 6.3 implies that every separable preference matrix on Q corresponds to a linear extension of the partial order $\succeq_D$ on $\mathcal{P}(Q)$. Proposition 6.2 implies that every separable preference matrix has the mirror property. But, as the next example shows, these necessary conditions are not sufficient.

Example 6.4. Note: Only the upper half of R_6 is given; the lower half is assumed completed according to the mirror property.

$$R_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ & & \vdots & & \end{pmatrix}$$

It can be verified directly that R_6 is a linear extension of $\succeq_D$. Obviously, it has the mirror property. But R_6 is not separable, as is clear because $11010 \succ 10111$ (inducing an order of $100 \succ 011$ on $\{2, 3, 5\}$) while $10101 \succ 11000$ (inducing the opposite order on $\{2, 3, 5\}$).

The preference matrix R_6 is nonseparable because it lacks a crucial property that we call consistency. For example, the two relations $11010 \succ 10111$ and $10101 \succ 11000$ are inconsistent. We next formulate this property precisely.

Definition 6.5. Let $>$ be a linear order on $\mathcal{P}(Q)$ and let $A, B \subseteq Q$ with $A > B$. Then $>$ is *consistent for* A *and* B iff, for every $S \subseteq A \cap B$, $A - S > B - S$. Also, $>$ is *consistent* iff $>$ is consistent for every $A, B \subseteq Q$ with $A > B$.

In Example 6.4, R_6 is not consistent for $A = \{1, 2, 4\}$ and $B = \{1, 3, 4, 5\}$ because $A > B$ but, if $S = \{4\}$, $B - S > A - S$. The next proposition shows that consistency is necessary and sufficient for separability.

Proposition 6.6. *A linear order $>$ on $\mathcal{P}(Q)$ is consistent if and only if it corresponds to a separable preference matrix.*

Proof. Let $>$ be a linear order on $\mathcal{P}(Q)$ and let P and $\succ$ be the corresponding preference matrix and linear order on X_Q. Suppose that P is not separable. Then there exists a subset S of Q and alternatives $y, z \in X_S$ and $y', z' \in X_{-S}$ such that $(y, y') \succ (z, y')$ and $(z, z') \succ (y, z')$. Let $A = \{q \in Q : y_q = 1 \text{ or } y'_q = 1 \text{ or } z'_q = 1\}$, $B = \{q \in Q : z_q = 1 \text{ or } y'_q = 1 \text{ or } z'_q = 1\}$, $S_1 = \{q \in -S : z'_q = 1 \text{ and } y'_q = 0\}$ and $S_2 = \{q \in -S : y'_q = 1 \text{ and } z'_q = 0\}$. Note that $S_1, S_2 \subseteq A \cap B$. It can be verified directly that $A - S_1$ represents (y, y') and that $B - S_1$ represents (z, y'), so $A - S_1 > B - S_1$. Likewise, $B - S_2 > A - S_2$ because $A - S_2$ represents (y, z') and $B - S_2$ represents (z, z'). It follows that, whether $A > B$ or $B > A$, $>$ is not consistent.

For the converse, suppose that $>$ is not consistent. Then there exist subsets A and B of Q and $S_0 \subset A \cap B$ such that $A > B$ but $B - S_0 > A - S_0$. Let $S = Q - S_0$. We show that $\succ$ is not separable on S. Define $Y = A - S_0$ and $Y' = B - S_0$, and note that $Y, Y' \subseteq S$. Also, define $Z = S_0$ and $Z' = \emptyset$, and note that $Z, Z' \subseteq -S$. From the hypotheses, $Y \cup Z = A > B = Y' \cup Z$ and $Y' \cup Z' = B - S_0 > A - S_0 = Y \cup Z'$. Thus, if y, y', z and z' are the elements of X_Q represented by Y, Y', Z, and Z', respectively, then $y, y' \in X_S$, $z, z' \in X_{-S}$, $(y, z) \succ (y', z)$, and $(y', z') \succ (y, z')$. Thus, P is not separable. $\square$

Combining the necessary conditions of Propositions 6.2 and 6.3 with the necessary and sufficient condition of Proposition 6.6 provides a means of generating all separable preference matrices on a given set of criteria, Q. Begin by constructing $(\mathcal{P}(Q), \succeq_D)$. Then form a linear extension of $\succeq_D$ by adding rows. As each row is added, check the corresponding subset of Q for consistency with the previous rows. As soon as half of the matrix has been generated, apply the mirror property to complete the remaining half.

We now apply this procedure to the cases $n = 2, 3, 4$. First (see Figure 1), when $n = 2$, $\succeq_D$ is a linear order, so only one separable preference matrix can be generated in this case. Thus,

$$P_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is the only normalized, canonically ordered, separable preference matrix on two criteria.

When $n = 3$, Figure 1 demonstrates that the first three rows of any separable matrix must be 111, 110, and 101. The fourth row could be 011 or 100, and both of these possibilities are consistent with the first three rows. The mirror property now implies that there are exactly two normalized, canonically ordered, separable preference matrices on 3 criteria,

$$P_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $n = 4$, a similar analysis yields 14 separable preference matrices, which are summarized in Figure 2. Following any path from 1111 to 0000 produces a separable preference matrix, and all separable preference matrices can be generated in this way.

Figure 2: Separable preference matrices on four criteria

We now return to the question of additivity. Recall from Example 5.6 and the discussion following it that separable preference matrices need not be additive for $n \geq 5$. We will now prove that separable preference matrices must be additive if $n \leq 4$. Our proof is based on Fishburn's Theorem (1970, Theorem 4.1C) that an ordering is not additive iff it admits a non-trivial Fishburn pair.

Definition 6.7. Let $\succ$ be a linear order on X_Q. A *Fishburn pair* for $\succ$ is (M, M'), where M and M' are $r \times n$ matrices with $r \geq 2$ such that the rows of M and M' are elements of X_Q, each column of M' is a permutation of the corresponding column of M, and $r_i \succeq r'_i$ for $i = 1, 2, \ldots, r$, where r_i and r'_i are the i^{th} rows of M and M', respectively. A Fishburn pair (M, M') is *trivial* iff $M = M'$.

In our discussion of Example 5.6, we exhibited a non-trivial Fishburn pair with $r = 4$ and $n = 5$.

Let S be a subset of the rows of a preference matrix P. A $\succ$-*filter for* S is a linear function on S, $L(r) = \sum_{i=1}^{n} c_i r_i$ that respects $\succ$, i.e. if $r \succ s$, then $L(r) \geq L(s)$. Such a filter will sometimes be denoted $\langle c_1, \ldots, c_n \rangle$. Our proof that every Fishburn pair is trivial if $n \leq 4$ is based on the following lemma:

Lemma 6.8. *Suppose that (M, M') is a Fishburn pair, that S is a subset of the rows of P with $\succ$-filter $L(\cdot)$, and that all rows of (M, M') are of the form (r_i, r'_i) where $r_i \in S$ and $r'_i \in S$. For every such i, $L(r_i) = L(r'_i)$.*

Proof. For rows from S, $r_i \succeq r'_i$, so $L(r_i) \geq L(r'_i)$. By Definition 6.7, there are equal numbers of 1's and 0's in corresponding columns of M and M', so that $\sum_i L(r_i) = \sum_i L(r'_i)$. If $L(r_k) > L(r'_k)$ for any k, this equation fails. Therefore, all rows (r_i, r'_i) of (M, M') must satisfy $L(r_i) = L(r'_i)$. $\square$

Note that conclusion of Lemma 6.8 remains true when there exist several subsets $S_1, S_2, \ldots, S_n$ of P such that $L(\cdot)$ is a $\succ$-filter on each S_j and the rows of (M, M') are all of the form (r_i, r'_i), where r_i, $r'_i \in S_j$ for some j.

Note also that Lemma 6.8, when applied with a suitable filter, places restrictions on the rows that can appear in a Fishburn pair for P. For instance, consider P_1, the unique separable ordering for $n = 2$. Applying the P_1-filter $L_1 = \langle 1, 1 \rangle$ yields $L_1(1,1) = 2, L_1(1,0) = L_1(0,1) = 1, L_1(0,0) = 0$. Lemma 6.8 then implies that, in any row (r, r') of any Fishburn pair, either $r = r'$ or $r = (1,0)$ and $r' = (0,1)$. Thus, $L_1(\cdot)$ produces a partition $\mathcal{P} = \{\{(1,1)\}, \{(1,0),(0,1)\}, \{(0,0)\}\}$ of the rows of P_1 such that, in any Fishburn pair, all rows are of the form (r, r'), where both r and r' belong to the same $S \in \mathcal{P}$.

Subsequent filters can be used to further refine this partition. For instance, $L_2 = \langle 1, 0 \rangle$ is a $\succ$-filter on each $S \in \mathcal{P}$. But $L_2(1,0) = 1$ and $L_2(0,1) = 0$, and so, applying Lemma 6.8 again, we see that no Fishburn pair can contain the row $((1,0),(0,1))$. It follows from our earlier application of $L_1(\cdot)$ that each Fishburn pair is trivial. Hence, P_1 is additive.

This process can be easily generalized and applied to higher dimensions. The task of proving the additivity of an order P is thus reduced to that of finding an appropriate sequence of filters, that, when applied with Lemma 6.8, produce a partition of the rows of P into singleton sets. The proof of the next proposition formalizes this process.

Proposition 6.9. *Let $|Q| \leq 4$, where $X_q = \{0,1\}$ for each $q \in Q$, and let $\succ$ be a linear order on X_Q with preference matrix P. If $\succ$ is separable, then $\succ$ is additive.*

Proof. Suppose that $L(\cdot)$ is a $\succ$-filter on P. A Fishburn pair (M, M') can be thought of as a collection of pairs of rows of P, i.e. elements of $P \times P$. By Lemma 6.8, these pairs of rows (r,s) must satisfy $L(r) = L(s)$. It follows that P can be partitioned into subsets of the form $S_x = \{r \in P : L(r) = x\}$ for $x \in \mathbb{R}$, such that (M, M') contains only pairs of rows from $S_x \times S_x$ for some x.

Moreover, Lemma 6.8 implies that this partition of P may be further refined by applying a $\succ$-filter on S_x for any x. We will show that, for all orders satisfying the hypotheses of the proposition, it is possible to find a sequence of $\succ$-filters that *completely refines* P, i.e. partitions the rows of P into one-element subsets. If so, any Fishburn pair consists only of rows of the form (s,s), and is therefore trivial.

The proof of the $n = 2$ case appears above. For $n = 3$, consider first the separable ordering P_2. The filter $L_1 = \langle 1,0,0\rangle$ shows that any row (r,s) of any Fishburn pair satisfies $r_1 = s_1$. The set of rows r satisfying $r_1 = 1$ can be completely refined using the two filters $L_2 = \langle 1,1,1\rangle$ and $L_3 = \langle 1,1,0\rangle$. The same two filters completely refine the set of rows r satisfying $r_1 = 0$, completing the proof for P_2. For P_3, we can apply the filter $L_1 = \langle 1,1,1\rangle$ to show that, if $r \neq s$ in any row of any Fishburn pair, then $r \succ s$ and r and s are chosen from either $(1,1,0) \succ (1,0,1) \succ (0,1,1)$ or from $(1,0,0) \succ (0,1,0) \succ (0,0,1)$. In either case, the filters $L_2 = \langle 1,0,0\rangle$ followed by $L_3 = \langle 0,1,0\rangle$ now completely refine P_3, completing the proof in the $n = 3$ case.

For $n = 4$, the 14 separable preference orderings are shown in Figure 2. Number them 1 to 14 from the left. It can be verified that the sequences of filters given in the following table completely refine the corresponding preference matrices, thus completing the proof.

Ordering	$\mathbf{L_1}$	$\mathbf{L_2}$	$\mathbf{L_3}$	$\mathbf{L_4}$
1	$\langle 1,1,1,0\rangle$	$\langle 1,1,0,0\rangle$	$\langle 1,0,0,0\rangle$	$\langle 0,0,0,1\rangle$
2	$\langle 1,1,0,0\rangle$	$\langle 1,0,1,0\rangle$	$\langle 0,0,0,1\rangle$	$\langle 1,1,1,0\rangle$
3	$\langle 1,1,0,0\rangle$	$\langle 1,0,1,0\rangle$	$\langle 0,0,0,1\rangle$	$\langle 1,1,1,0\rangle$
4	$\langle 1,0,0,0\rangle$	$\langle 1,1,0,0\rangle$	$\langle 1,1,1,1\rangle$	$\langle 1,1,1,0\rangle$
5	$\langle 1,1,1,0\rangle$	$\langle 1,1,0,0\rangle$	$\langle 0,0,0,1\rangle$	$\langle 1,0,0,0\rangle$
6	$\langle 1,1,0,0\rangle$	$\langle 1,1,1,1\rangle$	$\langle 1,0,1,0\rangle$	$\langle 1,0,0,0\rangle$
7	$\langle 1,1,1,0\rangle$	$\langle 1,0,0,0\rangle$	$\langle 0,0,0,1\rangle$	$\langle 1,1,0,0\rangle$
8	$\langle 2,1,1,1\rangle$	$\langle 1,1,0,0\rangle$	$\langle 1,1,1,0\rangle$	$\langle 1,0,0,0\rangle$
9	$\langle 2,1,1,1\rangle$	$\langle 1,1,0,0\rangle$	$\langle 1,0,1,0\rangle$	$\langle 1,0,0,0\rangle$
10	$\langle 1,0,0,0\rangle$	$\langle 1,1,1,1\rangle$	$\langle 1,1,0,0\rangle$	$\langle 1,0,1,0\rangle$
11	$\langle 2,1,1,1\rangle$	$\langle 1,1,0,0\rangle$	$\langle 1,1,1,0\rangle$	$\langle 0,0,0,1\rangle$
12	$\langle 1,1,1,0\rangle$	$\langle 1,0,0,1\rangle$	$\langle 1,1,0,0\rangle$	$\langle 0,0,0,1\rangle$
13	$\langle 2,1,1,1\rangle$	$\langle 1,1,1,0\rangle$	$\langle 0,1,1,0\rangle$	$\langle 0,0,0,1\rangle$
14	$\langle 1,1,1,0\rangle$	$\langle 1,0,0,1\rangle$	$\langle 0,0,0,1\rangle$	$\langle 1,1,0,0\rangle$

□

As a comment on the proof of Proposition 6.9, we remark that separability is a similar condition to additivity on a linear order over multidimensional binary alternatives, but separability is weaker. For instance, in Example 5.6 we displayed an ordering that is separable but not additive. We noted that, according to this ordering,

$$A_0 = \{2,3,5\} \succ B_0 = \{1,4\}$$
$$C_0 = \{1,5\} \succ D_0 = \{2,3\}.$$

But notice that $A_0 \cup C_0 \not\succ B_0 \cup D_0$, since it follows from Proposition 6.3 that

$$B_0 \cup D_0 = \{1,2,3,4\} \succ A_0 \cup C_0 = \{1,2,3,5\}$$

for any normalized, canonically ordered separable ordering. (Note that the order from Example 5.6 is indeed normalized and canonically ordered.) Thus an ordering can be separable but not be consistent with set conjunction, an analogue of addition.

In fact, separability does imply addition-like properties provided appropriate non-intersection conditions are met. For an order $\succ$ that is separable on all subsets of Q, the following statements are implied by Proposition 6.6:

- if $A, B, C \subseteq Q$ with $C \subseteq A \cap B$, then $A \succ B$ iff $A - C \succ B - C$;
- if $A, B, C \subseteq Q$ with $C \cap (A \cup B) = \emptyset$, then $A \succ B$ iff $A \cup C \succ B \cup C$;
- if $A, B, C, D \subseteq Q$ with A, B, C, and D all disjoint, $A \succ B$, and $C \succ D$, then $A \cup C \succ B \cup C \succ B \cup D$.

The last property is particularly illuminating. Note that it is analogous to the arithmetic relation $a > b$ and $c > d$ implies $a + c > b + d$. But it is also the property that failed for A_0, B_0, C_0, and D_0, above (where $C_0 \cap A_0 \neq \emptyset$ and $B_0 \cap C_0 \neq \emptyset$).

Compared to additivity, therefore, separability is a similar but weaker restriction on an ordering. For orderings on small sets, the differences are not important; there is relatively little flexibility in choosing a separable ordering. But as the size of the set of criteria increases, the "combinatorial explosion" in the number of possible orderings allows many separable orderings, not all of which consistent with the stronger condition of additivity.

We conclude this section by exhibiting explicit linear additive values for all possible separable linear orders on X_Q where $Q = \{1, \ldots, n\}$ and $n \leq 4$. It is easy to verify directly that

- $v(x_1, x_2) = 2x_1 + x_2$ is a value function for P_1;
- $v(x_1, x_2, x_3) = 4x_1 + 2x_2 + x_3$ is a value function for P_2; and
- $v(x_1, x_2, x_3) = 2x_1 + 1.5x_2 + x_3$ is a value function for P_3.

For $n = 4$, the following table gives the coefficients of a linear value function $v(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} c_i x_i$ for each of the fourteen separable preference matrices represented in Figure 2. Recall that the preference matrices are numbered 1 through 14, beginning on the left in Figure 2.

Ordering	$\mathbf{c_1}$	$\mathbf{c_2}$	$\mathbf{c_3}$	$\mathbf{c_4}$
1	7.34	5.56	3.36	1
2	7.70	5.20	3.00	1
3	8.03	4.87	2.67	1
4	8.57	4.33	2.13	1
5	6.70	6.20	2.76	1
6	6.78	6.13	1.42	1
7	6.44	4.55	4.05	1
8	8.10	4.55	4.05	1
9	9.10	4.55	4.05	1
10	10.92	4.55	4.05	1
11	2.90	2.10	1.50	1
12	2.82	2.25	1.68	1
13	2.65	1.85	1.75	1
14	2.70	2.30	1.89	1

7 Summary and Conclusions

Separable preferences over multidimensional alternatives should be easy to understand in the sense that they induce well-defined orderings on every subset of the dimensions, orderings that are independent of the outcomes on all other criteria. We have examined the properties of such preferences when alternatives are binary, which is the simplest case of discrete alternatives. The discrete setting is a natural model for multiple elections, and the binary case for the special case of simultaneous referenda. In contrast, continuous spaces of alternatives are the norm in economic models.

Our results indicate important differences in the properties of separability in discrete versus continuous settings. In the continuous context, separability is a very strong property—under suitable regularity conditions, it is equivalent to additivity. But Gorman's theorems never apply in the discrete case, where separability imposes much weaker restrictions on an ordering.

Our investigation was based on a very general definition of preference separability, not only on individual criteria but also on sets of criteria. We also demonstrated useful properties of separability (the mirror property, the relation to the lattice $(\mathcal{P}(Q), \succ_D)$, and consistency), and used them to establish a method for constructing separable preference matrices. We exhibited all separable preference matrices on four or fewer criteria, and gave linear additive value functions for each one.

Many interesting questions remain. For instance, further research might address additional restrictions in the discrete case that would guarantee the strong properties of separability that Gorman demonstrated in the continuous case. Another issue is the extent to which Gorman's Theorems fail in the discrete case. Though closure under intersections is the only one of Gorman's conclusions that we confirmed, a more thorough investigation of set-theoretic properties of separability might find other general properties of separable discrete preferences.

Finally, we noted above that separability is similar to additivity, but weaker. In our view, the equivalence of these properties when alternative sets are small is an artifact of the restrictions imposed by this very smallness. Therefore we think it unlikely that a direct proof that separability implies additivity when there are four or fewer alternatives will be found.

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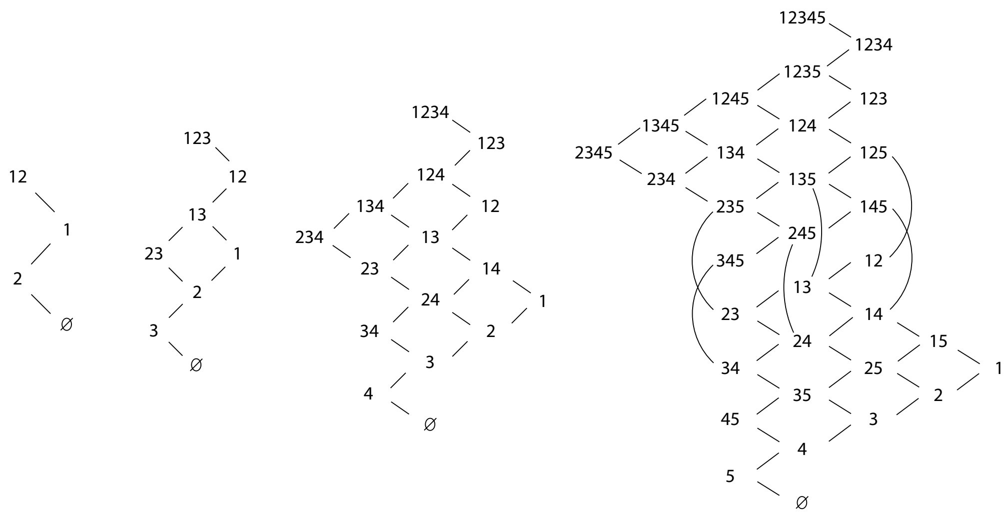
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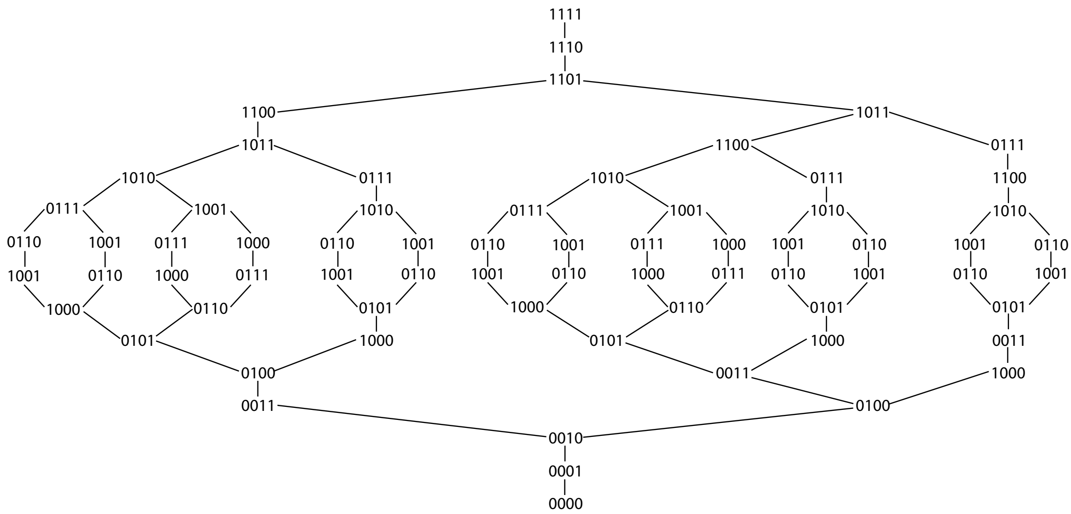
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