Homogeneous Sub-Finsler 3-Manifolds and their Geodesics

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Smooth Manifolds

The earth is a sphere, but locally, it looks like a plane. This is because the earth is so large and our field of vision too small to capture the **curvature** of the earth. This phenomena is essential to the idea of a **manifold**: a space in which if you "zoom in" at any point, it will look like a point, a line, a plane, or an Euclidean space of dimension three or higher. A **curve** on a manifold is simply a path, e.g. a river or highway.

For our project, we focused on a type of manifold called sub-Finsler manifolds where one may measure distances and also do calculus on it.

Metrics on Finsler Manifolds

Metrics are what we use to measure distance. If we're on the surface of the earth, we could say the distance from Moscow and Sydney is the length of the straight line going through the earth's interior. However, we are probably more interested in traveling on the surface, without tunneling. Then, what we are interested in is a **geodesic**, a minimal path with respect to the metric, the way we are measuring. In the case of spheres, geodesics are great circles such as the equator.

On a **Finsler** manifold, we have a metric F, sometimes called a Lagrangian function, which gives us information about **vectors** on the **tangent spaces** of a manifold. We use the following length functional to calculate length of curves:

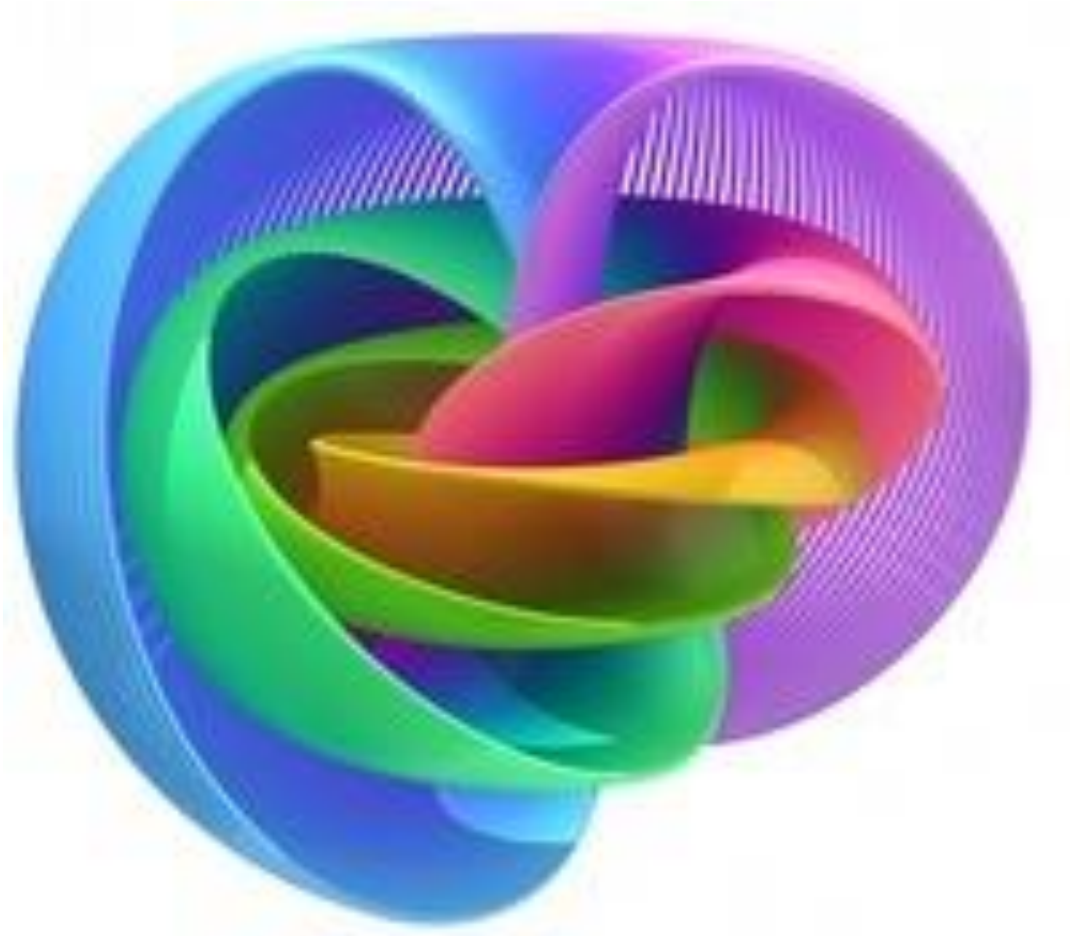
$$\mathscr{L}(\gamma) = \int_a^b F(\dot{\gamma}(t))\,dt$$

Sub-Finsler Manifolds

On a manifold, instead of studying just the tangent spaces at each point, we may go a step further and study **subspaces** of the tangent spaces. Hence, a sub-Finsler manifold is a triple: the manifold, the subspaces, and a metric.

Classification Using Lie Algebras

The geometry of these manifolds, even in low dimensions such as three, can become quite complicated.



Hopf Fibration of a 3-Sphere

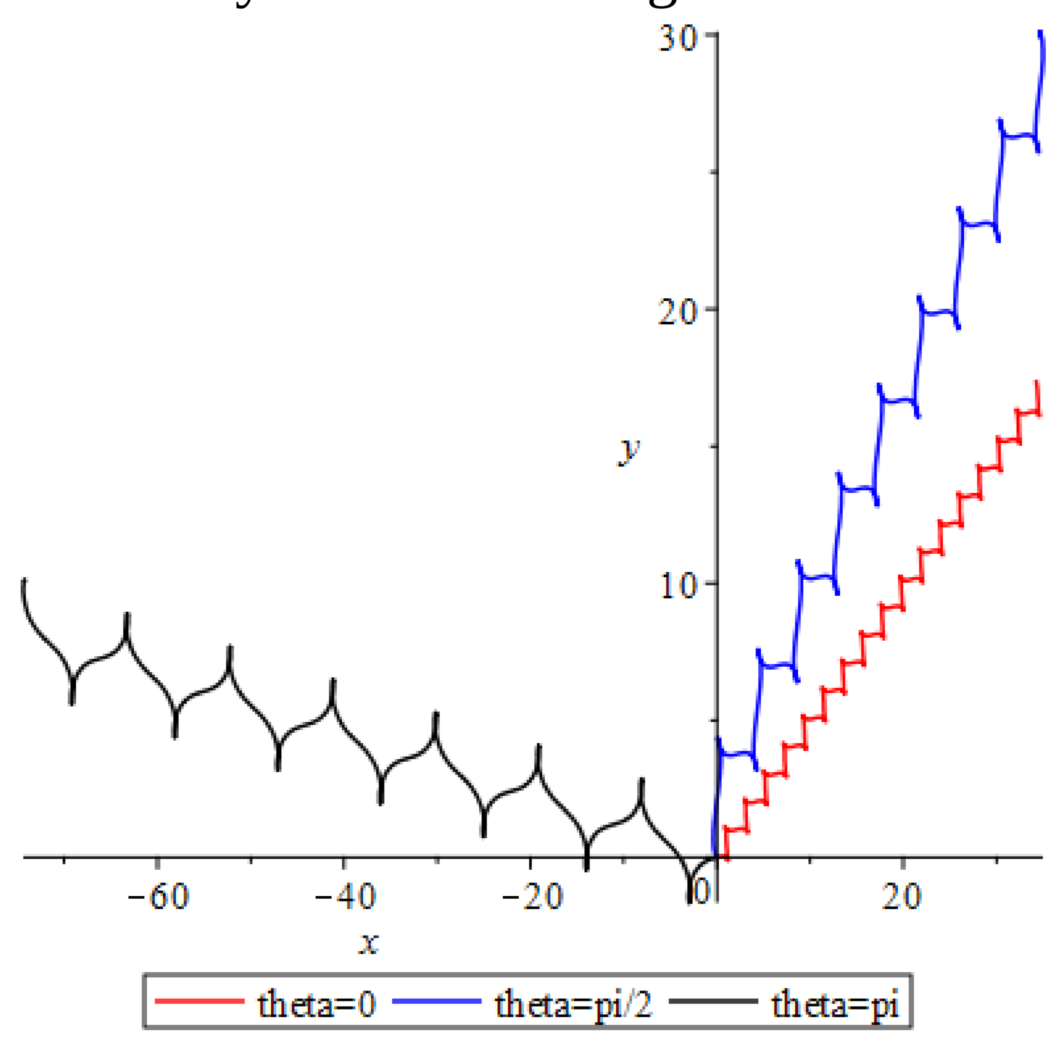
Thus, in order to understand the geometry of the manifold, we studied their corresponding **Lie algebras**, a vector space with a bracket operator.

Method

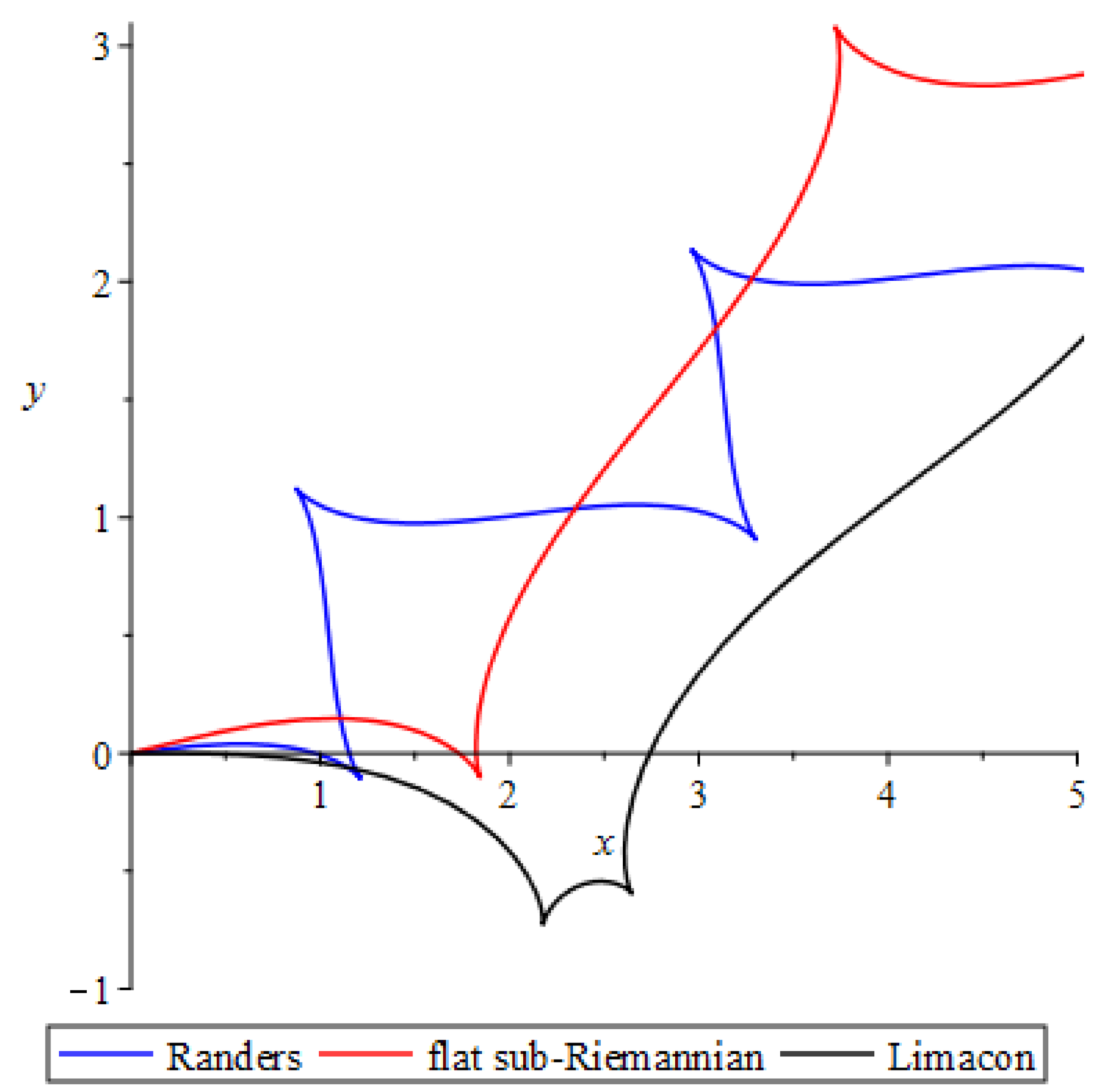
With the results from Šnobl and Winternitz [1], we took known families of Lie algebras and, through the aid of Maple, adapted their **framings** to match the Lie algebra corresponding to the manifold of interest. In doing so, we classified the manifold.

Results

We classified all 3-dimensional sub-Finsler manifolds via Lie algebras by completing K. Hughen's [2] partial classification on sub-Riemannian manifolds and then used Clelland and Moseley's [3] work to "translate" to the sub-Finsler case. We also computed some differential equations which provide the necessary conditions for geodesics.



Geodesics for Randers Metric on E(2)



The projected D-curves (geodesics) for E(2) with different metrics.

Application to Control Theory

One motivation for wanting to classify these manifolds comes from **control theory.** Take a system, say, a car's cruise control. Optimizing fuel efficiency, time spent driving, among other controls, makes the problem complex. We may "translate" the problem into geometric terms, solve the problem there, and come back with a solution for our problem. Classifying sub-Finsler manifolds allows us to solve whole family of problems where each problem is essentially the same problem as another but in disguise.

Future Research

Classifying homogeneous manifolds via Lie algebras may be extended to manifolds of higher dimension with 4-manifolds being the natural next step. Higher dimension manifolds are often studied in physics but we may also extend the applications to more complicated problems in control theory.

References

[1] L. Šnobl and P. Winternitz..,Classication and Identication of Lie Algebras.

[2] K. Hughen, The Geometry of Sub-Riemannian Three Manifolds.

[3] J.N. Clelland and C.G. Moseley. Sub-Finsler geometry in dimension three.

Image: Wikipedia (Hopf fibration)

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