HOMOGENEOUS SUB-FINSLER 3-MANIFOLDS AND THEIR GEODESICS

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In [1], Clelland and Moseley investigated the local geometry of sub-Finsler 3-manifolds, a natural generalization of sub-Riemannian 3-manifolds. One motivation for the generalization comes from control theory. Given a particular system, such as cruise control for a car, the configuration of this system can be represented by elements of a *state space*, the set of possible configurations. The restrictions on this system give rise to the nature of the space itself, thus changing its geometry.

For our project, we first completed a proof by Hughen of absorbing torsion to produce a smaller G-structure. With this, we then classified all the *Lie algebras* ("Lee") corresponding to homogeneous contact 3-manifolds with 3-dimensional symmetry groups and homogeneous sub-Finsler metrics by completing K. Hughen's partial classification of the sub-Riemannian analog [2]. With the classification completed, we then computed the differential equations necessary for a given curve to be a *geodesic*, a generalization of "shortest path" on a manifold.

Intuitively, a manifold M is a space which appears locally Euclidean. *Lie groups* are a type of manifold with the property of being *homogeneous*, i.e. acts transitively on itself. Furthermore, we have continuity and differentiability on Lie groups which gives them a *differentiable* manifold structure. The differentiable and algebraic structures are interesting in their own right; for our project, we studied Lie groups with a third requirement: a metric structure. The Riemannian metric, a positive definite tensor $\langle \cdot, \cdot \rangle = \varphi^2 + \eta^2$ where φ, η are differentiable 1-forms, acts within the tangent spaces of our manifold and may change smoothly between tangent spaces. The notion of a sub-Riemannian manifold is a generalization of a Riemannian manifold: simply allow the metric to act on a *distribution D*, a collection of subspaces of fixed dimension of the tangent spaces. A curve $\gamma : I \to M$ tangent to D is called a D-curve, analog to a geodesic. In order to find the length of γ, we use the length functional $\mathscr{L}$ which is $\int \langle \gamma', \gamma' \rangle^{1/2} \, dt$. Since we also wish to find a geodesic, we seek critical points of $\mathscr{L}$. This corresponds to an optimization within control theory and finding equivalence classes of problems.

Generalizing further, we come to a *Finsler metric $F : TM \to [0, \infty)$* where TM is the tangent bundle on M. We require F to be regular, positively homogeneous, and strongly convex. A Riemannian metric is a special case of a Finsler metric in which F is given by the square root of a quadratic form. The length of curve γ is a similar integral to the sub-Riemannian case. Sub-Finsler geometry is yet another generalization; triples (M, D, F), with the metric acting on a distribution, are the objects of study.

In order to complete the classification, we consulted Šnobl and Winternitz [3] for all families of Lie algebras. A Lie algebra $\mathfrak{g}$ is the tangent vector space at the identity of a Lie group with a bracket operator. After calculating the structure equations of a given manifold, we adapted the framings of known 3-dimensional Lie algebras to see if they have the same structure and properties as the Lie algebra corresponding to our manifold, e.g. nilpotence, solvability. We were able to complete Hughen's classification in the sub-Riemannian case and thus, by following Clelland and Moseley [1], classify the sub-Finsler case as well. We also calculated some of the differential equations as necessary conditions for geodesics under various metrics.

Throughout our research, we relied heavily on results from calculus of variations, linear algebra, group theory, differential geometry, and even differential topology. We also used the software Maple additionally with the Cartan-Kahler package, thanks to J. Clelland, to aid us in our computations.

The natural next step is to move to higher dimensions. However, because of dimensionality and constraints, there are manifolds on which curves behave rigidly yet are not geodesics. In order to bypass this problem in our calculations, we will need to perform a projection of critical curves from an unconstrained space onto the manifold of interest.

This project is an excellent foretaste of the content of graduate school and research and a paper is forthcoming. I've also come to better understand what Courant and Robbins [4] meant: "Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. Though different traditions may emphasize different aspects, it is only the interplay of these antithetic forces and the struggle for their synthesis that constitute the life, usefulness, and supreme value of mathematical science."

References

[1] J.N. Clelland and C.G. Moseley. Sub-Finsler Geometry in Dimension Three. *Differential Geometry and its Applications*, 24(6):628–651, 2006.

[2] K. Hughen. The Geometry of Sub-Riemannian Three Manifolds. *Duke University*, (unpublished dissertation), 1995.

[3] L. Šnobl and P. Winternitz. *Classification and Identification of Lie Algebras.* AMS, Providence: Rhode Island, 2014.

[4] Richard Courant and Herbert Robbins. *What is Mathematics?* Oxford University Press, New York, 1941.