Record #1855

Michael Bolt Sabbatical Application 2009-2010

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Project Title The Möbius geometry of hypersurfaces.

Project Abstract The area of the proposed research is several complex variables. One of the guiding principles in this area is to understand how geometry and analysis work together. Specfically, the proposed research is focused on four problems whose solutions will help to explain connections
between geometry and analysis in a Mobius invariant setting.
The primary object of study is a hypersurface in complex Euclidean space. Essential to the study of a hypersurface is the notion of curvature. This measures the amount of bending of the hypersurface from what is considered at. The notion of curvature is easily generalized to other geometries and, in particular, to the study of hypersurfaces under complex analytic transformations. In this setting, the relevant curvatures are encoded in a matrix called the Levi form. Another setting is the study of hypersurfaces under M obius
transformations of the ambient Euclidean space. Again, there are curvatures encoded in a matrix|the matrix is analogous to the Levi form. The goal of the proposed project is to identify new connections between these notions of curvature and to use them to better understand certain basic structures in several complex variables.

Outside Funding Efforts to obtain funding: 1. Fall, 2007: I applied for a Calvin Research Fellowship for the project “Möbius geometry of submanifolds with applications”. [Approved] 2. Fall, 2006: I applied for a standard research grant from the National Science Foundation for the project “RUI: Geometric estimates for complex analysis and applications”. [Approved] 3. Fall, 2006: I applied for a Calvin Research Fellowship for the project “Geometric estimates for complex analysis.” [Approved] 4. Fall, 2005: I applied for a travel grant from the American Mathematical Society and National Science Foundation in order to attend and speak at the International Congress of Mathematicians. [Approved] 5. Fall, 2004: I applied for a standard research grant from the National Science Foundation for the project “RUI: Inversive geometry with applications to complex analysis, and applications to conformal mapping”. [Rejected] 6. Fall, 2004, I applied for a Calvin Research Fellowship for the project “Invariant arclength from distance functions”. [Approved]

Project Details Bolt pd.pdf see page 3

Project Outcome The focus of the proposed project is to develop new methods in differential geometry and to apply them to problems in several complex variables. I expect to publish my work in a peer-reviewed mathematics research journal. I also expect to present my work at conferences and in seminars at nearby research universities. For instance, I am frequently invited to speak in special sessions at the regional meetings of the American Mathematical Society. I am also frequently invited to speak in departmental seminars at research universities, for instance, at the University of Michigan, University of Illinois, etc. These audiences are able to provide me with critical feedback that can help to advance my scholarship program.

The emphasis of the project will be to generate new mathematics. In my recent work I have helped to uncover the Möbius invariance of certain objects studied in several complex variables. (These invariance properties were not previously known.) Until now, I have been working to clarify these properties and to establish rigidity results for them. So far my work has resulted in three publications. The proposed work will be to draw further connections

	between the invariant objects and to connect my recent work to recent work of other mathematicians. Except for regular visits to Ann Arbor, the proposed research will be done at Calvin. The project will therefore directly impact the research environment at Calvin. Students at Calvin will benefit from the project indirectly in knowing that mathematics is an active research field. Furthermore, the sabbatical period will enable me to concentrate on my advanced scholarship during the academic year. In the summers, I can then devote more of my time to working with research students on projects that are accessible to them. These students will therefore benefit directly from the project.
Pedagogical Impact	The project would strengthen the mathematical research environment at Calvin. It is desirable to have mathematicians at Calvin who are engaged in both teaching and research---these areas support each other in important ways. It is also important for Calvin students to know that mathematics is a dynamic field that continues to be driven by challenges that arise from science and technology. The opportunity to pursue the project during the academic year would enable me to be fully engaged in working with students during summer months on NSF-funded projects. The proposed project is not appropriate for students and is only indirectly connected to the other projects on which I will be working with students.
Itinerary / Work Schedule	At the very beginning of the sabbatical period I plan to apply for a renewal of my NSF grant. Throughout the sabbatical period, I will work on the four proposed problems. I expect to make regular visits to the University of Michigan so that I can meet with my collaborator and mentor, David Barrett. When at Michigan, I also expect to participate in the Seminar in Several Complex Variables and Dynamics and the RTG working seminar in SCV and Complex Dynamics.
Budget	There are no additional expenses or costs associated with the project. (Travel expenses connected with my visits to Ann Arbor, for example, are covered by the NSF grant.)
Research Funding History	Research funding history: 1. Awarded a two-course CRF for Spring, 2009, for the project “Möbius geometry of submanifolds with applications”. 2. Awarded full funding ($89,351) from the NSF for the project “RUI: Geometric estimates for complex analysis and applications”. This is a 3-year award that includes research and travel support for students. 3. Received travel funding from the NSF ($1300) and Brazil's National Council for Scientific and Technological Development ($800) to attend and speak at the 4th Workshop on Geometric Analysis of PDE and Several Complex Variables, in Serra Negra, Brazil, August 2007. 4. Awarded a two-course CRF for Spring, 2008, for the project “Geometric estimates for complex analysis”. 5. Awarded travel funding from the American Mathematical Society and NSF ($2150) to attend and speak at the International Congress of Mathematicians in Madrid, August 2006. 6. *Awarded a one-course CRF for Spring, 2006, for the project “Invariant arclength from distance functions”. *After holding the CRF in spring of 2006, I applied for a 3-year NSF grant in fall of 2006, and I was subsequently awarded that grant in summer of 2007. I intend to apply for a renewal of the grant at the very beginning of the sabbatical period, in fall of 2009.

Owner: Crook, Dawn
Created: NOV-04-2008 8:55 AM (EST)
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SABBATICAL PROJECT DESCRIPTION

MICHAEL BOLT

The goal of the proposed project is to develop methods of classical differential geometry and to apply them to mathematical structures that arise in several complex variables. In order, I will briefly describe the following:

1) the geometry of hypersurfaces in Euclidean space
2) an introduction to several complex variables
3) work already done to connect these subjects
4) four problems I will pursue during the sabbatical period.

The sabbatical period will be used for pursuing the four problems.

During the sabbatical, I expect to make frequent visits to the University of Michigan so that I can meet with my collaborator and mentor, David Barrett. Barrett is one of the leading researchers in the field and is currently working on problems close to the ones I propose. At Michigan, I expect to also participate in the Seminar in Several Complex Variables and Dynamics and the RTG working seminar in SCV and Complex Dynamics.

Regarding a timeline, I expect that with the release from teaching during a semester and interim, I can make significant progress on two of the problems I propose. (It is doing the mathematics that requires the concentrated time commitment.) My work should result in a publication in a peer-reviewed mathematics research journal. I also expect to present my work at conferences and in seminars at nearby research universities.

Geometry of hypersurfaces in Euclidean space

Hypersurfaces in Euclidean space are a generalization to higher dimensions of the more familiar surfaces in space. The connection is as follows. A surface is essentially a 2-dimensional object inside an ambient 3-dimensional space. The resulting codimension is $3 - 2 = 1$. Hypersurfaces are essentially $(n - 1)$-dimensional objects within an ambient n-dimensional space. They, too, have codimension 1.

Mathematically, these surfaces behave alike in that there is just one missing direction, the normal direction, and this is the common tool that is used for doing analysis on these objects.

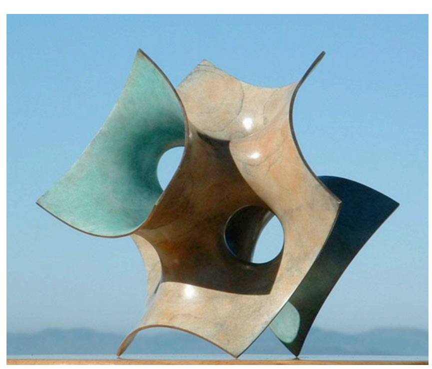

Figure 1. *Genus-2 Costa Surface in a Cube*, by Carlo Séquin (2005)

Date: September 20, 2008.

The surface in Figure 1 is an example of a complete minimal surface that was discovered by the Brazilian mathematician Celso Costa in 1982. This surface extends indefinitely—making it complete—and it has the property that if it is bumped in any direction, then its local surface area becomes larger—making it minimal. The latter property illustrates the kind of rigid intuitions to which a differential geometer is inclined.

The key to understanding surfaces in Euclidean space is the notion of curvature. This is illustrated in Figure 2. The left-hand side of the figure shows a surface M together with its tangent plane and normal direction at point p. The right-hand side shows the curvature at

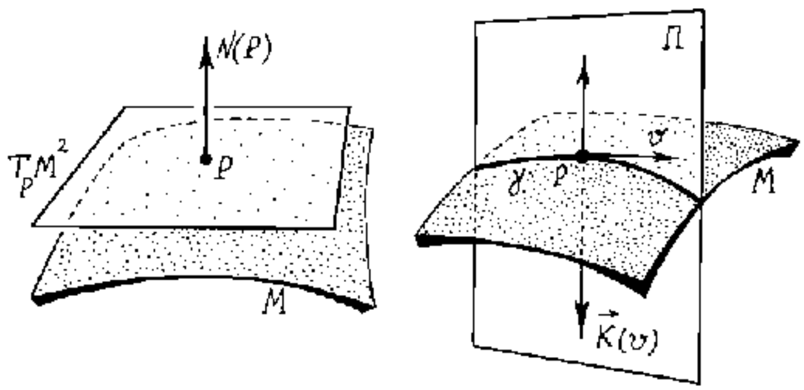

FIGURE 2. Tangent space and normal direction for a surface in space

point p in the tangential direction v. The curvature is computed by intersecting the surface with a plane containing the normal and tangential directions, and by measuring how the resulting curve γ bends away from the tangential direction.

What makes the underlying geometry Euclidean is the fact that the curvatures are preserved by the Euclidean motions of the ambient space—translations, rotations, and reflections. Spheres play a special role in this geometry since their curvature is constant. That is, the amount of bending is the same at each point of the surface and when measured with respect to any tangential direction.

(In fact, it is the Euclidean geometry that one studies in high school. The notion of congruence, for instance, is unaffected by translation, rotation, or reflection.)

The main structural equations for a surface are the equations of Gauss and Codazzi. They behave as follows. If one has complete information about the curvatures at all points of a surface (i.e., a kind of 'scorecard' at each point of the surface), then the Gauss-Codazzi equations explain the interactions that are needed in order for these curvatures to piece together to determine a realizable surface.

I have used the Gauss-Codazzi equations extensively in three of my recent papers, [3, 4, 5]. In these papers I characterize hypersurfaces according to properties of certain structures arising in several complex variables. In part, my results indicate the extent to which these structures are in fact Euclidean.

To finish, it is worth mentioning that the codimension 1 objects, namely the hypersurfaces, are a natural reference point for studying objects of greater codimension—the general submanifolds of Euclidean space. We return to this in the last section.

INTRODUCTION TO SEVERAL COMPLEX VARIABLES

Rather simply, complex analysis is the study of the calculus of a complex variable, where 'complex' refers to the complex numbers. That is, include in your number system the imaginary unit, i, where $i^2 = -1$. This subject originated in 1799 with the discovery by Carl Friedrich Gauss that the complex numbers form an algebraically closed field—that is, any polynomial can be completely factored over the complex numbers. This fact has been extremely useful for many areas of mathematics.

It was the subsequent physical applications, however, that established complex analysis as a valuable subject in its own right. The important feature of the complex variable is that it provides a tool for constructing a conformal map, that is, a map that preserves angles. As an example, Figure 3 illustrates the Joukowski transformation, which is a conformal map from the exterior of a disc to the exterior of a cross-section of an airfoil. The streamlines for

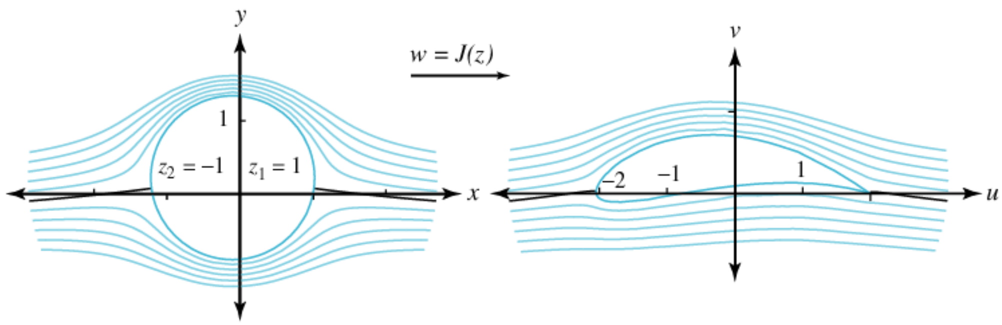

Figure 3. (a) Flow around a disc (b) Flow around the airfoil

the flow around the disc can be readily computed, and these streamlines can then be pushed forward, using the conformal map, in order to find the flow around the airfoil. Furthermore, by using the transformation, one can modify the shape of the airfoil and compute, for instance, the effect on the aerodynamic lift.

The study of conformal maps is usually done inside the class of holomorphic functions—as maps, these functions are most everywhere conformal, and they can be expressed locally using power series, a topic familiar to first year calculus students. This partially motivates the subject of several complex variables, because in higher dimensions, the holomorphic functions are exactly the ones that can be expressed locally using power series. A key difference, however, is that general holomorphic functions no longer preserve angles.

It is worth mentioning that a single complex variable corresponds with two real dimensions. For instance, the complex number $z = x + iy$ corresponds with the point (x, y). So already, two complex variables corresponds with four real dimensions, etc.

The first really striking differences between one and several complex variables were uncovered by Friedrich Hartogs in a series of results from the early 1900's [9, 10]. While studying domains in several variables, he uncovered a geometric condition that permits all the holomorphic functions on a domain to be extended to a larger domain. (Simple examples show this is impossible in one variable.) Subsequent work by E. E. Levi led to a formulation of the Levi problem, the natural converse statement to the Hartogs results [12].

The Levi problem persisted as a great mathematical challenge until the 1950's, when Kiyoshi Oka discovered the method of plurisubharmonic functions, and he used it to solve the Levi problem in the affirmative [13]. The essential geometric notion is pseudoconvexity—this is a holomorphic version of convexity and can be described using a subset of the curvatures of the hypersurface that bounds the domain.

Already in the work of Hartogs and Levi, it was clear that geometry plays a critical role in the analysis of several complex variables. Conversely, Oka's work shows how analysis in several complex variables places restrictions on the geometry of the underlying domain. This interplay between geometry and analysis has had applications in many areas of mathematics and has been a guiding principle in several complex variables ever since [14].

Möbius geometry of hypersurfaces

A person studying Euclidean geometry considers objects congruent if they differ only by a translation, rotation, and possible reflection. Similarly, a person studying complex analysis considers objects congruent if they differ only by a biholomorphism. The biholomorphisms include the Euclidean motions, but they include other kinds of transformations, too. Between these classes of transformations are the Möbius transformations. In one dimension and higher, Möbius transformations are characterized by the property that they preserve angles on the full ambient space.

To illustrate, Figure 4 shows a domain coloring representation for a one-dimensional Möbius transformation. The example shows a map that sends the point $(0, 0)$ to the point

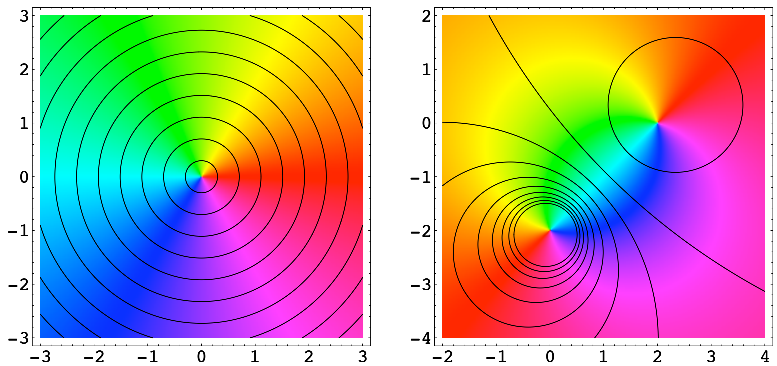

Figure 4. Domain coloring representation for a Möbius transformation

$(2, 0)$ and the point at infinity to the point $(0, -2)$. Notice that it also maps circles on the left-hand side to circles on the right-hand side.

As described earlier, the curvatures for a hypersurface in Euclidean space are preserved by Euclidean motion. In several complex variables, just a subset of these curvatures, encoded in the Levi form, are preserved by biholomorphism. Since Möbius transformations include Euclidean motions, but are included in the biholomorphisms, it is plausible that there are additional curvatures, besides those in the Levi form, that are preserved by Möbius transformation. I identified these extra curvatures in a recent paper [3]. There I also use the Gauss-Codazzi equations to characterize surfaces for which the curvatures vanish.

More recently I characterized the surfaces for which the curvatures are (in a sense) constant [4]. So far, that result holds only for complex dimension two.

In earlier work, I showed that the Leray-Aĭzenberg transform is Möbius invariant [6]. This transform is just one among many of the higher-dimensional analogues of the Cauchy transform from the study of one complex variable. The Möbius invariance, however, makes it a leading candidate for the one closest to the Cauchy transform, as the Cauchy transform is likewise Möbius invariant.

Problems concerning the Möbius geometry of hypersurfaces

For my sabbatical, I propose to investigate the following problems. With a release from teaching for a semester and interim, I expect to make significant progress on two of them.

Problem 1. *In dimensions greater than two, characterize the surfaces that have constant Möbius curvature.*

In [3], I characterized the surfaces for which the Möbius curvatures are identically zero—they are exactly the hermitian quadrics. The result holds in dimensions greater than one. In [4], I characterized in dimension two the surfaces with constant Möbius curvature. If the magnitude of the curvature is not one, these surfaces are images of

$$M_\varepsilon = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Real}(z_1 + \varepsilon z_2^2) + |z_2|^2 = 0\}.$$

(If the magnitude is one, it appears they might have to be the so-called tube domains.) So far, I have a reasonable formulation for what it means for curvatures to be constant in high dimensions, and I have identified a class of surfaces for which these curvatures are constant.

Problem 2. *In dimension two, compute the spectrum of the Leray-Aĭzenberg transform for the surface M_ε defined above ($|\varepsilon| \neq 0, 1$).*

As mentioned earlier, I proved in [6] that the Leray-Aĭzenberg transform is Möbius invariant for surfaces in dimensions greater than one. (This is when the transform is defined with respect to the Fefferman measure.) Moreover, I characterized in [4] the surfaces in dimension two for which the Möbius invariant curvatures are in a sense constant. It is expected that the geometric information of the hypersurface should be reflected by the analytic data.

Barrett and Lanzani already considered this problem for convex Reinhardt domains in two dimensions [2]. The problem here is to compute the spectrum on a surface exhibiting a special kind of symmetry—one for which the connection to Möbius curvature should be more explicit. The proposed problem will hopefully be a collaborative project with Barrett.

Problem 3. *For general surfaces in dimension two, estimate the norm of the Leray-Aĭzenberg transform using only Möbius invariant quantities.*

This problem is related to the previous one, but rather than looking for detailed analytic information for a particular surface, the goal is it to estimate the norm of the transform for surfaces of general type. The estimates might use the Möbius invariant curvature, they might use other invariant quantities, or they may use some combination of them. The general theme will be to draw connections between analytic and geometric structures from a Möbius perspective. This project extends previous collaborative work with Barrett for curves in dimension one [1].

Problem 4. *Determine any role played by the Möbius invariant curvatures with respect to immersions of CR submanifolds into higher dimensional spheres.*

At a recent conference, Peter Ebenfelt (from UCSD) asked if it would be possible to extend my work on hypersurfaces to the situation of general submanifolds in complex Euclidean space. This question is motivated by his recent work with Huang and Zaitsev concerning immersions of CR submanifolds into higher-dimensional spheres [7, 8]. The expectation is that Möbius invariants of submanifolds might indicate obstructions to the existence of immersions. Conversely, a positive result could help provide a new analogue to the Whitney embedding theorem in complex analysis.

For high-codimension submanifolds, there are multiple normal directions, so curvatures will be encoded in a matrix-valued function. (I expect the curvatures can be characterized in a manner analogous to Robert Hermann's characterization of the Levi form [11].) The problem then is to also express these curvatures algebraically, and to connect them to the work of Ebenfelt, et al.

References

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