Homotopy of Manifolds

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Even though you know the earth is a sphere, the evidence around you suggests that you live on a flat, 2-dimensional plane. This is because a sphere is a 2-manifold; it looks like a plane from the perspective of something very small relative to the size of the sphere. The universe is a 3-manifold. We move around in a way that gives the impression of 3-dimensional space from our perspective, but we are really moving within 4-dimensional space, including time as another dimension. This generalizes to the concept of an n-manifold, an object which is locally n-dimensional. Our research focused on categorizing certain $n$-manifolds.

What if you lived on giant torus (a doughnut-shaped surface)? A torus is another 2-manifold, it looks like a plane from the perspective of a small person walking on the surface. How would you be able to tell that you did not live on a sphere? Imagine you explore with a piece of rope. On a sphere, if you leave a rope trail, come back to where you started, and gather your rope in, it will always come back to you. However, if you live on a torus, and you walk around the handle, grab the other end of your rope, and pull tight, the rope gets caught on the handle. It doesn’t come back in.

Mathematics captures this difference between the sphere and the torus in with the concept of homotopy groups. Every space, like a sphere or torus, has associated homotopy groups, which are invariant. This means that every space that is the same has the same homotopy groups. This is useful in distinguishing between spaces.

Returning to our example, the torus has a loop that doesn’t contract, but every loop on the sphere contracts. Because of this, the torus has a different first homotopy group than the sphere. Since for the spaces to be the same, they would need to have the same homotopy groups, they cant be the same.

An important mathematical object we worked with was a bouquet of spheres, named for the resemblance to a bouquet of flowers. To form a bouquet, take some number of spheres and join them all at a single point. We used a different bouquet of spheres in order to better understand each of the different $n$-manifolds.

We explored connections between homotopy groups of a bouquet of spheres to answer the question of whether certain manifolds are the same or different. We used matrices to represent transformations between different manifolds and were able to say, based on the structure of a given matrix, whether two manifolds are equivalent with respect to homotopy.

Because of last year’s experience, I became more familiar with the mathematical objects being studied this year. I investigated more deeply into the change-of-basis matrices for the Hilton basis of higher dimension cell complexes, and I am thankful for the result I was able to discover. When pondering on the concepts in question, I often felt like being thrown into a wilderness of the unknowns, because I was facing a completely new topic and had no assurance of whether I would be able to produce some significant result. However, an important lesson I learned is to courageously enter the wilderness and start the exploration while fully acknowledging the possibility of fruitlessness. For several times unexpected discoveries came out in seemingly hopeless situations. This unexpectedness makes research in mathematics both frightening and exciting for me.