Homotopy of Manifolds

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August 1, 2014

Even though you know the earth is a sphere, the evidence around you suggests that you live on a flat, 2-dimensional plane. This is because a sphere is a 2-manifold; it looks like a plane from the perspective of something very small relative to the size of the sphere. The universe is a 3-manifold. We move around in a way that gives the impression of 3-dimensional space from our perspective, but we are really moving within 4-dimensional space, including time as another dimension. This generalizes to the concept of an n-manifold, an object which is locally n-dimensional. Our research focused on categorizing certain n-manifolds.

What if you lived on a giant torus (a doughnut-shaped surface)? A torus is another 2-manifold, it looks like a plane from the perspective of a small person walking on the surface. How would you be able to tell that you did not live on a sphere? Imagine you explore with a piece of rope. On a sphere, if you leave a rope trail, come back to where you started, and gather your rope in, it will always come back to you. However, if you live on a torus, and you walk around the handle, grab the other end of your rope, and pull tight, the rope gets caught on the handle. It doesn’t come back in.

Mathematics captures this difference between the sphere and the torus in with the concept of homotopy groups. Every space, like a sphere or torus, has associated homotopy groups, which are invariant. This means that every space that is the same has the same homotopy groups. This is useful in distinguishing between spaces.

Returning to our example, the torus has a loop that doesn’t contract, but every loop on the sphere contracts. Because of this, the torus has a different first homotopy group than the sphere. Since for the spaces to be the same, they would need to have the same homotopy groups, they can’t be the same.

An important mathematical object we worked with was a bouquet of spheres, named for the resemblance to a bouquet of flowers. To form a bouquet, take some number of spheres and join them all at a single point. We used a different bouquet of spheres in order to better understand each of the different n-manifolds.

We explored connections between homotopy groups of a bouquet of spheres to answer the question of whether certain manifolds are the same or different. We used matrices to represent transformations between different manifolds and were able to say, based on the structure of a given matrix, whether two manifolds are equivalent with respect to homotopy.

In comparison to last year, I enjoy doing research more this summer. For one thing, this year I have more realistic expectation about the outcomes of our research. I no longer expect to produce something ground-breaking but rather appreciate the small things that we are able to prove. Moreover, after taking several proof-based math courses over the past two semesters, I have developed more appreciation and patience for proofs, and this makes research more fun for me. Finally, I learned to deal with seemingly hopeless situations. Sometimes the problem I am trying to tackle seems too deep and complicated for me to handle. However, I found that if I force myself working on it, something interesting and meaningful, and often times unexpected will happen. Sometimes the problem is solved in a way different than I thought, and other times a new problem that worth tackling emerges. Overall, I am thankful for the research opportunity this summer and feel more prepared for works in graduate schools.