THE KREIN MATRIX AND POLYNOMIAL EIGENVALUE PROBLEMS

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The goal for the summer of 2013 was to analyze a modified form of the eigenvalue problem using a specific formulation. The problem involves matrices, rectangular arrays of numbers arranged in rows and columns, and vectors, geometric objects with direction and magnitude. The familiar form of this particular problem is $A\vec{x} = \lambda\vec{x}$. $A$ is an $n \times n$ matrix and $I_n$ is the identity matrix ($AI_n = I_nA = A$). $\lambda$ is a constant called the eigenvalue. Given a matrix, we find a vector that when multiplied with $A$, will return a multiple of the vector. The quadratic problem is an expansion of this linear problem, taking the form $P(\lambda) = M_0 + \lambda M_1 + \lambda^2 I_n$. We look for similar results. One benefit to understanding the properties of these eigenvalues is its applications to differential equations. Such equations primarily model population growth, finances, or principles within physics and engineering (more applications can be found in Tisseur and Meerbergen’s paper [1]). In general, methods for finding solutions to any large matrix equation is not difficult: numerical computation has reduced the task to merely writing a computer program. However, those methods only find the values themselves with no further information. In applications, knowing other properties of the eigenvalue is also important as it predicts long-term behaviors based on small changes of initial conditions. An alternative is to use what is known as the Krein formulation which creates a function that clearly represents each part of the matrices. This shows how they behave and contribute to the final result. The Krein formulation also assigns signatures to each eigenvalue which reveals stability, quantity, and location in the complex plane. This is especially important because as the matrices get larger, numerical computation requires more time.

In order to understand the quadratic problem, the Krein formulation was first applied to the simpler linear problem (an $n \times n$ matrix $B$ replaces $I_n$). Without losing the properties of the original matrix, the original matrices can be pre-processed and manipulated using mathematical tools to make a simplified equivalent problem. This processing also allows us to choose a $k$ value which represents the number of negative eigenvalues of $B$. This $k$ determines the size of the new problem. So if large matrices are being used at first, the new problem can always be made smaller. However, there is some difficulty in representing matrices graphically. Thus, a method is borrowed from Richard Kollár and Peter Miller’s paper [2]. The problem $A\vec{x} = \lambda I_n\vec{x}$ can be rewritten as $(A - \lambda I_n)\vec{x} = 0$. Kollár and Miller then showed that this problem could be remade into another eigenvalue problem. This seems counterintuitive at first but we are transforming the problem into something we are more familiar with solving. The final expression is: $P(\lambda)\vec{v}(\lambda) = \mu(\lambda)\vec{v}(\lambda)$ where $P(\lambda)$ equals $A - \lambda B$ and $\mu$ is an eigenvalue for $P(\lambda)$. Since all the terms depend on $\lambda$, when $\mu = 0$, the original problem is solved (recall the standard: $(A - \lambda I_n)\vec{x} = 0$). This result yields $n$ curves which may cross the line $\mu = 0$ up to $n$ times. The sign of the slope (i.e. derivative) of the Krein function at these roots gives information concerning the stability of the eigenvalues. Techniques to solving the quadratic problem resemble the methods previously discussed.

One of the interesting results found concerns the linear problem. The result is called the Interlacing Theorem. It was originally described by Horn and Johnson [3] using a linear algebra proof. The theorem states that the eigenvalues of the standard problem ($B = I_n$) and of the general problem alternate. This theorem was proven using numerics and data plotting. Other results found are more deeply rooted within advanced mathematical theory and will not be discussed here.

This research experience has given me a glimpse of how current mathematical research is done. There is a fair mix of theory and abstract based work combined with computation and computer programming which I have not had much exposure to. My proficiency in various areas have improved, particularly in using software such as MATLAB and Wolfram Mathematica. This experience has shown me the broad contexts of applied mathematics which I had little experience with before. It also showed me how interwoven the various branches of mathematics truly are. I have enjoyed this period simply because of my interest in mathematics as well as experiencing how mathematics is discovered.

References