Rigidity of Frameworks and Laman’s Theorem

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1 Background

The focus of our research was Laman’s Theorem. There have been several geometric proofs of Laman’s Theorem, but our objective was to construct an algebraic proof. The theorem deals with frameworks and rigidity. A framework is essentially a figure in the plane made up of an interconnected series of bars that are allowed to rotate at the connecting joints. A framework is rigid if there is no "flexibility" or "tilting" in the framework. Laman’s theorem gives conditions for the minimum number and organization of edges required for a framework to be rigid.

2 Research Methods

We began by studying Ideals, Varieties, and Algorithms by Cox, Little, and O’Shea. Next, we used the distance formula to form polynomials describing the edges of a framework. Using Mathematica, we calculated Groebner bases of these polynomials for both rigid and non-rigid frameworks and studied the results. We were unable to compute Groebner bases for larger frameworks, as Groebner bases take exponential time to compute. We then switched gears to a linear algebra approach, where we described the edges of the framework with standard-form line equations. We formed a matrix with these equations and explored the properties of the matrix, both before and after Gaussian elimination.

3 Results

We came up with several conjectures and theorems about the ideals and Groebner bases of rigid frameworks, in an attempt to prove Laman’s Theorem using the properties of the Groebner basis. We also made progress towards an algebraic proof of Laman’s Theorem using two different approaches: defining the edges with both quadratic and linear equations, and observing those results.

4 Personal Comments

From this project, we learned that the problem of rigidity is much deeper than it first appears. Although we did not locate the complete algebraic proof of Laman’s Theorem, we pursued several methods that might eventually lead to a valid proof. The work we did has also improved our skills in mathematics, including logic, geometry, and algebra.